The semantics of algebraic quantum mechanics and the role of model theory.

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B.Zilber, *The semantics of the canonical commutation relations*
arxiv.org/abs/1604.07745

Geometric dualities

Affine commutative $\mathbb{C}$-algebra

$$R = \mathbb{C}[X_1, \ldots, X_n]/I$$

Complex algebraic variety

$$\mathbf{V}_R$$

Commutative unital C^*-algebra

$$A$$

Compact topological space

$$\mathbf{V}_A$$

Affine reduced k-algebra

$$R = k[X_1, \ldots, X_n]/I$$

The geometry of k-definable points, curves etc of an algebraic variety $\mathbf{V}_R$

...

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Why model theory?

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In general the syntax may come with a topology! (as in C^*-algebras).

Zariski geometries as geometric semantics

The structure $\mathbf{V} = (V, L)$ with a topology on its cartesian powers is said to be (Noetherian) Zariski if it satisfies

- Closed subsets of V^n are exactly those which are L-positive-quantifier-free definable.
- The projection of a closed set is quantifier-free definable (positive quantifier-elimination).
- A *good* dimension notion on closed subsets is given.
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Theorem. *Noetherian Zariski geometries allow elimination of quantifiers and are stable of finite Morley rank.*

Further geometric dualities

Affine commutative $\mathbb{C}$-algebra R	Complex algebraic variety $\mathbf{V}_R$
Commutative C^*-algebra A	Compact topological space $\mathbf{V}_A$
Affine reduced k-algebra R	The k-definable structure on an algebraic variety $\mathbf{V}_R$
$*$-algebra A at roots of unity	Zariski geometry $\mathbf{V}_A$
Weyl-Heisenberg algebra $\langle Q, P : \; QP - PQ = i\hbar \rangle$	

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A noncommutative duality Theorem

For the category of algebras "at roots of unity" there is an equivalence of categories

$$A_{\mathbf{V}} \longleftrightarrow \mathbf{V}_A.$$

$A_{\mathbf{V}}$ – co-ordinate algebra, $\mathbf{V}_A$ – Zariski geometry.

A non-commutative example "at root of unity"

Non-commutative 2-torus $\mathbf{V}_A$ at $\epsilon = e^{2\pi i \frac{m}{N}}$
has co-ordinate ring $A =$
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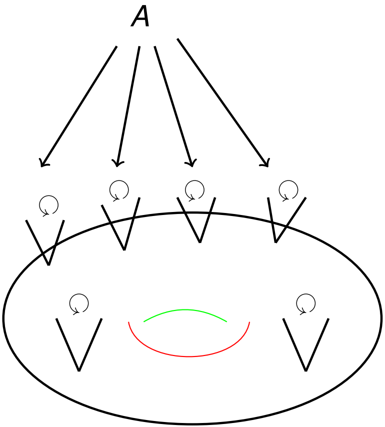
Points α on the torus have structure of an N-dim Hilbert space V_α with a distinguished system of **canonical orthonormal bases**



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On suggestion of Weyl and following Stone – von Neumann Theorem replace the Weyl-Heisenberg algebra by the category of **Weyl $*$-algebras**

$$A_{a,b} = \left\langle U^a, V^b : \ U^a V^b = e^{2\pi iab} V^b U^a \right\rangle, \quad a, b \in \mathbb{R}.$$

Think:

$$U^a = e^{iaQ}, \ V^b = e^{\frac{2\pi}{\hbar} ibP}.$$

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$QP - PQ = i\hbar$. Correcting the syntax

When $a, b \in \mathbb{Q}$ the algebra $A_{a,b}$ is at root of unity. We call such algebras **rational Weyl algebras.**
Ignore the non-rational ones. Replace "the algebra given by $QP - PQ = i\hbar$" by the category $\mathcal{A}_{\mathrm{fin}}$ of rational Weyl algebras

$$A_{a,b} = \left\langle U^a, V^b : \; U^a V^b = e^{2\pi i ab} V^b U^a \right\rangle, \quad a, b \in \mathbb{Q}$$

with morphisms = embeddings.

Categories $\mathcal{A}_{\rm fin}$ and $\mathcal{V}_{\rm fin}$

Note:

$$A_{a,b} \hookrightarrow A_{c,d} \text{ iff } \exists n, m \in \mathbb{Z} \;\; cn = a \,\&\, dm = b$$

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$$\mathbf{V}_{A_{a,b}} \to \mathbf{V}_{A_{c,d}}$$

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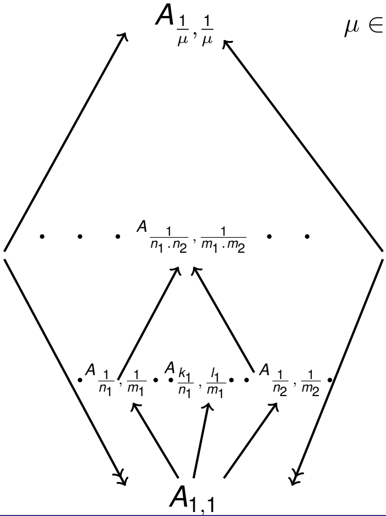
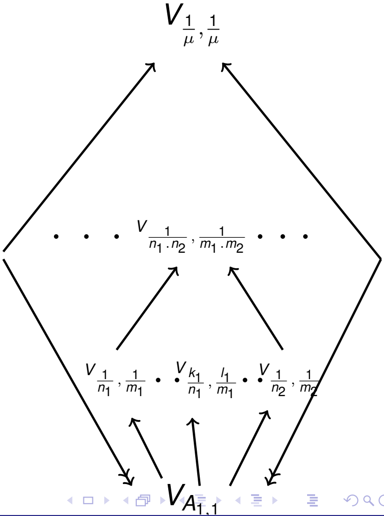
Note: $\mathbf{V}_{A_{a,b}}$ is interpretable in $\mathbf{V}_{A_{c,d}}$ but not the other way round.

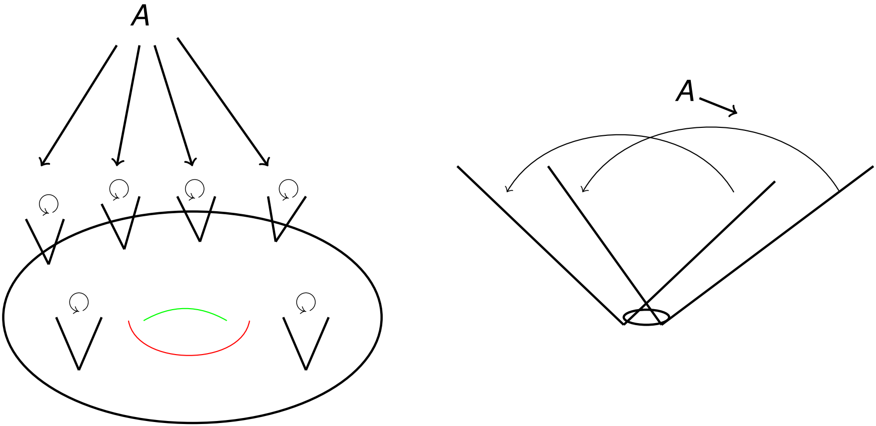
The duality functor $A \mapsto \mathbf{V}_A$ can be interpreted as defining a **sheaf of Zariski geometries** over the lattice $\mathcal{A}_{\text{fin}}$

$\mu \in {}^*\mathbb{Z}$





How noncommutative $\mathbf{V}_{A_{\frac{1}{m},\frac{1}{n}}}$ deforms into $\mathbf{V}_{A_{\frac{1}{\mu},\frac{1}{\mu}}}$



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The subalgebra of operators which *survive the limit*

$$A_* \subset A_{\frac{1}{\mu},\frac{1}{\mu}}$$

acts on the substructure

$$\mathbf{V}_* \subset \mathbf{V}_{A_{\frac{1}{\mu},\frac{1}{\mu}}}$$

which *survive the limit*.

The space of states $\mathbb{S}$.

The structure $\mathbb{S}$ is a homomorphic image of $\mathbf{V}_*$ under a homomorphism called lim,

$$\lim : \mathbf{V}_* \to \mathbb{S}, \quad {}^*\mathbb{Q} \to \mathbb{R}.$$

This can also be classified as a generalisation of the

- standard part map,
- specialisation,
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$\mathbb{S}$ *is a symplectic space with a vector field and Fourier transforms on it.*

See e.g. G. Lion and M.Vergne, **The Weil Representation, Maslov Index, and Theta Series** Birkhauser 1980

Operators acting on $\mathbb{S}$

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We define (interdefinably) in each member $\mathbf{V}_{a,b}$ of the ultraproduct:

$$\mathrm{Q} := \frac{U^a - U^{-a}}{2ia}, \quad \mathrm{P} := \frac{V^b - V^{-b}}{2ib}$$

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Then for any vector e of norm 1,

$$(\mathrm{QP} - \mathrm{PQ})e = i\hbar e + (s_1 - s_2)$$

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Under the lim $s_1 - s_2$ vanishes!

So, **in the space of states:** $\mathrm{QP} - \mathrm{PQ} = i\hbar I.$

Observables

A relation, a function or an operator which is defined on the multisorted structure $\mathcal{V}_{\mathrm{fin}}$ is said to be **observable** if it is respected by lim and the image in $\mathbb{S}$ is non-trivial.
In particular, **an observable relation is Zariski closed.**

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Examples.

- Operators P and Q.
- $|\langle \mathbf{w}_1 | \mathbf{w}_2 \rangle|_{\mathrm{Dir}} := \mu \cdot |\langle \mathbf{w}_1 | \mathbf{w}_2 \rangle|$, renormalised probability.
- ...

Gauss quadratic sums survive the limit

$$\sum_{n=0}^{N-1} e^{2\pi i \frac{n^2}{N}} = \mathrm{e}^{-i\frac{\pi}{4}}\sqrt{N}$$

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Here, for $a = \frac{k}{m}$ it is crucial that μ is divisible by k.

Example of calculation. Quantum harmonic oscillator.

The Hamiltonian:

$$H = \frac{1}{2}(\mathrm{P}^2 + \mathrm{Q}^2)$$

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The time evolution operator :

$$K^t = K_H^t := e^{-i\frac{H}{\hbar}t}, \quad t \in \mathbb{R}.$$

This “induces” the automorphism of the category of algebras

$$U^a \mapsto e^{-\frac{2\pi a^2 \sin t \cos t}{2}} U^{a \sin t} V^{a \cos t}$$
$$V^a \mapsto e^{\frac{2\pi a^2 \sin t \cos t}{2}} U^{-a \cos t} V^{a \sin t}$$

(in $\mathbf{V}_*$ we only consider t such that $\sin t, \cos t \in \mathbb{Q} - \{0\}$).

Example. Quantum harmonic oscillator.

Write $|x\rangle$ for eigenvectors of Q with eigenvalues $x \in \mathbb{R}$.
Then the *kernel of the Feynman propagator* is calculated in $\lim \mathbf{V}_*$ as

$$\langle x_1 | K^t x_2 \rangle_{\mathrm{Dir}} = \sqrt{\frac{1}{2\pi i \hbar \sin t}} \exp i \frac{(x_1^2 + x_2^2)\cos t - 2x_1 x_2}{2\hbar \sin t}.$$

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The trace of K^t,

$$\mathrm{Tr}(K^t) = \int_{\mathbb{R}} \langle x | K^t x \rangle = \frac{1}{\sin \frac{t}{2}}.$$

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Note that in terms of conventional mathematical physics we have calculated

$$\mathrm{Tr}(K^t) = \sum_{n=0}^{\infty} e^{-it(n+\frac{1}{2})},$$

a non-convergent infinite sum.

An analogy: p-adic and motivic integration

$$\int_{A(\mathfrak{P})} |f(z)|^t dz = g(q,t)$$

where $\mathfrak{P}$ is a locally compact non-archimedean field, $q = p^n$ is the cardinality of the residue field of $\mathfrak{P}$, $t \in \mathbb{R}$ and g is a nice function which **does not depend on** $\mathfrak{P}$.

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In the formulae above x appears at any high enough level of $\mathbf{V}_{\frac{1}{m},\frac{1}{m}}$ of the category as

$$q = p^{n^2} = e^{ix^2}; \quad p = e^{\frac{2\pi i}{m^2}}$$

$$\langle x_1 | K^t x_2 \rangle_{\mathrm{Dir}} = \int_{\mathbb{R}} f(y)^t dy$$

$$g(q, t) = \sqrt{\frac{1}{2\pi i\hbar \sin t}} \exp i \frac{(x_1^2 + x_2^2) \cos t - 2x_1 x_2}{2\hbar \sin t}.$$

Conclusions

- The resulting semantics of the canonical commutation relation $QP - PQ = i\hbar$ suggests that the universe of quantum mechanics is a **huge finite** space of states.
- The known list of observables can be explained by the semantics.
- The calculations of key integrals can be reduced to calculations of finite sums without invoking continuous limits.