Basis problems in set theory

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Leeds, August 3, 2016

Outline

- What is a basis problem?
- Examples of basis problems
- Tukey classification problems
- Descriptive-theoretic basis results
- Basis results for *n*-gaps in $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$
- Using Forcing to solve a basis problem
- Basis results for compact spaces
- Ramsey basis problems
- Canonical relations
- Ramsey degrees
- Topological dynamics

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Definition

Given a quasi-ordered class $(\mathcal{K}, \leq)$ of (relational) structures of the same type, we say that $\mathcal{K}_0 \subseteq \mathcal{K}$ is a **basis** *of $\mathcal{K}$ if for every $K \in \mathcal{K}$ there is $K_0 \in \mathcal{K}_0$ such that $K_0 \leq K$.*

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Definition

Aronszajn ordering

is an uncountable linearly ordered set orthogonal to $\{\omega_1^*, \omega_1, \mathbb{R}\}$.

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PFA implies that the class of **Aronszajn orderings** *has basis* $\{C^*, C\}$, *where* C *is any uncountable linear ordering whose cartesian square is the union of countably many chains.*

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PFA implies that the class of **uncountable linear orderings** *has basis*

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where B *is any set of reals of cardinality* $\aleph_1$ *and where* C *is any uncountable linear ordering whose cartesian square is the union of countably many chains.*

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Trees

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Definition

For two trees S and T, by $S \leq_1 T$ we denote the fact that S can be **topologically embeded** *into T, i.e., that there is $f : S \to T$ which is* **strictly increasing** *and $\wedge$-preserving.*

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- *Assuming PFA the class $(\mathcal{A}, \leq)$ is generated by a discrete chain $\mathcal{L}$ of* **Lipschitz trees** *such that $(\mathcal{L}/\mathbb{Z}, \leq)$ is the $\aleph_2$-saturated linear ordering of cardinality $\aleph_2 = 2^{\aleph_1}$.*

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A partially ordered set P is **Tukey reducible** *to a partially ordered set Q, in notation $P \leq_T Q$, if there is a map $f : P \to Q$ that maps unbounded subsets of P to unbounded subsets of Q, or equivalently, a map $g : Q \to P$ which maps cofinal subsets of Q to cofinal subsets of P.*
When $P \leq_T Q$ and $Q \leq_T P$ we write $P \equiv_T Q$ and say that P and Q are **Tukey equivalent** *or* **cofinaly similar**.

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Remark

In the class of (upwards) **directed** *posets $P \equiv_T Q$ is equivalent to saying that P and Q are* **isomorphic** *to cofinal subsets of a single directed poset R.*

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Remark

If $\mathcal{U}$ is a P-point ultraflter on ω then $\mathcal{U} \not\equiv_T \mathcal{U}_{\max}$.

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Moreover, letting $D_0 = 1$, $D_1 = \omega$, $D_2 = \omega_1$, $D_3 = \omega \times \omega_1$, *and* $D_4 = [\omega_1]^{<\omega}$, *every* **partially ordered set** *of cardinality at most* $\aleph_1$ *is Tukey equivalent to one of these:*

- $\bigoplus_{i<5} n_i D_i$ $(i < 5, n_i < \omega)$,
- $\aleph_0 \cdot 1 \oplus \bigoplus_{i=2}^{4} n_i D_i$ $(2 \leq i < 5, n_i < \omega)$,
- $\aleph_0 \cdot \omega_1 \oplus n_4 [\omega_1]^{<\omega}$ $(n_4 < \omega)$,
- $\aleph_0 \cdot [\omega_1]^{<\omega}$,
- $\aleph_1 \cdot 1$.

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Remark

The topology of a basic order is uniquely determined by the order itself. It is the topology of sequential convergence where a sequence is set to be convergent if all of its subsequences have further subsequences that are bounded.

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- *D is compact iff $D \equiv_T 1$.*
- *If D is analytic and not locally compact then $\mathbb{N}^{\mathbb{N}} \leq_T D$.*

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Corollary

Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on ω such that $\mathcal{V}$ is a P-point . If $\mathcal{U} \leq \mathcal{V}$ then there is a **continuous map** $g : \mathcal{V} \to \mathcal{U}$ *witnessing this.*

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Corollary

P-point ultrafilters have no more than continuum many Tukey-predecessors.

Basis problem for n-gaps

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Notation:
Fix a countable index set N. For $a, b \subseteq N$, set

$$a \subseteq^* b \text{ iff } a \setminus b \text{ is finite,}$$

$$a \perp b \text{ iff } a \cap b \text{ is finite.}$$

For $\mathfrak{X}, \mathfrak{Y} \subseteq \mathcal{P}(N)$, set

$$\mathfrak{X} \perp \mathfrak{Y} \text{ iff } (\forall a \in \mathfrak{X})\,(\forall b \in \mathfrak{Y}) \;\; a \perp b.$$

$$\mathfrak{X}^\perp = \{b : (\forall a \in \mathfrak{X}) \;\; b \cap a \text{ is finite}\}.$$

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Let $\Gamma = \{\Gamma_i : i \in n\}$ be a n-sequence of preideals on the set N and let $\mathfrak{X}$ be a family of subsets of n.

1. We say that Γ is **separated** if there exist subsets $a_0, \ldots, a_{n-1} \subset N$ such that $\bigcap_{i \in n} a_i = \emptyset$ and $x \subset^* a_i$ for all $x \in \Gamma_i$, $i \in n$.

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2. We say that Γ is an $\mathfrak{X}$-**gap** if it is not separated, but $\bigcap_{i \in A} x_i =^* \emptyset$ whenever $x_i \in \Gamma_i$, $A \in \mathfrak{X}$.

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Definition

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When $\mathfrak{X} = \{n\}$ consists only of the total set $n = \{0, \ldots, n-1\}$, then an $\mathfrak{X}$-gap will be called an n_*-**gap**.

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The orthogonal of the gap Γ is $\Gamma^{\perp} = \left(\bigcup_{i \in n} \Gamma_i\right)^{\perp}$. The gap Γ is called **dense** if $\Gamma^{\perp}$ is just the family of finite subsets of N.

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For Γ and Δ two n_*-gaps on countable sets N and M, respectively, we say that

$$\Gamma \leq \Delta$$

if there exists a one-to-one map $\phi : N \to M$ such that for $i < n$,

1. if $x \in \Gamma_i$ then $\phi(x) \in \Delta_i$.
2. If $x \in \Gamma_i^{\perp}$ then $\phi(x) \in \Delta_i^{\perp}$.

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Two n_*-gaps Γ and Γ' are called **equivalent** if $\Gamma \leq \Gamma'$ and if $\Gamma' \leq \Gamma$.

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- *When* Γ *is a n-gap, the second condition can be substituted by saying that if* $x \in \Gamma^\perp$ *then* $\phi(x) \in \Delta^\perp$*.*

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- *Notice also that if* Δ *is a n-gap,* Γ *is a* n_**-gap, and* $\Gamma \leq \Delta$*, then* Γ *is an n-gap.*
- *Another observation is that the above definition implies that* $\phi(x) \in \Delta_i^{\perp\perp}$ *if and only if* $x \in \Gamma_i^{\perp\perp}$*, and* $\phi(x) \in \Delta^{\perp}$ *if and only if* $x \in \Gamma^{\perp}$*.*

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- *Notice also that if Δ is a n-gap, Γ is a n_*-gap, and $\Gamma \leq \Delta$, then Γ is an n-gap.*
- *Another observation is that the above definition implies that $\phi(x) \in \Delta_i^{\perp\perp}$ if and only if $x \in \Gamma_i^{\perp\perp}$, and $\phi(x) \in \Delta^{\perp}$ if and only if $x \in \Gamma^{\perp}$.*
- *Therefore the gaps $\{\Gamma_i^{\perp\perp} : i < n\}$ and $\{\Delta_i^{\perp\perp}|_{\phi``N} : i < n\}$ are completely identified under the bijection $\phi : N \to \phi``N$.*
- *There are other variants of the order $\leq$ between gaps but all lead essentially to the same theory.*

Remark

- *When* Γ *is a n-gap, the second condition can be substituted by saying that if* $x \in \Gamma^{\perp}$ *then* $\phi(x) \in \Delta^{\perp}$.
- *Notice also that if* Δ *is a n-gap,* Γ *is a* n_**-gap, and* $\Gamma \leq \Delta$, *then* Γ *is an n-gap.*
- *Another observation is that the above definition implies that* $\phi(x) \in \Delta_i^{\perp\perp}$ *if and only if* $x \in \Gamma_i^{\perp\perp}$, *and* $\phi(x) \in \Delta^{\perp}$ *if and only if* $x \in \Gamma^{\perp}$.
- *Therefore the gaps* $\{\Gamma_i^{\perp\perp} : i < n\}$ *and* $\{\Delta_i^{\perp\perp}|_{\phi``N} : i < n\}$ *are completely identified under the bijection* $\phi : N \to \phi``N$.
- *There are other variants of the order* $\leq$ *between gaps but all lead essentially to the same theory.*

Definition

An analytic n_*-gap Γ is said to be a **minimal analytic n_*-gap** if for every other analytic n_*-gap Δ, if $\Delta \leq \Gamma$, then $\Gamma \leq \Delta$.

Finite Basis Theorem for n-gaps

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Fix a natural number n. For every analytic n_*-gap Γ there exists a minimal analytic n_*-gap Δ such that $\Delta \leq \Gamma$.

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Up to permutations there is exactly 5 minimal analytic 2-gaps. Most of them already show up in the literature.

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Remark

Up to permutations there is exactly 5 minimal analytic 2-gaps. Most of them already show up in the literature.

Corollary

Let $\mathcal{U}$ be a countable family of pairwise disjoint analytic open subsets of $\beta\mathbb{N} \setminus \mathbb{N}$, and let $\{U_0, U_1, U_2\} \subseteq \mathcal{U}$ be such that $\overline{U_0} \cap \overline{U_1} \cap \overline{U_2} \neq \emptyset$. Then, there exists a point $x \in \overline{U_0} \cap \overline{U_1} \cap \overline{U_2}$ such that

$$|\{U \in \mathcal{U} : x \in \overline{U}\}| \leq 61.$$

Moreover, 61 is optimal in this result.

Compact sets of Baire-class-1 functions

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Theorem (T., 1999)

The class of non-metrizable **separable compact sets** *of Baire-class-1 functions defined on a Polish space has the 3-element basis*

$$\{S, D, P\},$$

where S is the split-interval, D the (separable version of the) Alexandrov duplicate of the Cantor set, and P the one-point compactification of the Cantor tree space.

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Remark

If x is a non-G_δ point of some compact set K of Baire-class-1 functions then K contains a topological copy of P where x plays the role of point at infinity.

Ramsey basis problems

Ramsey basis problems

Theorem (Ramsey 1930)

*For every natural number k and every relation $R \subseteq \mathbb{N}^k$ there is an infinite set $M \subseteq \mathbb{N}$ such that $R \upharpoonright M$ is $(\mathbb{N}, <)$-***canonical***.*

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Definition

*A relation $R \subseteq \mathbb{N}^k$ is $(\mathbb{N}, <)$-**canonical** on a set $M \subseteq \mathbb{N}$ if it is $\sim_{(\mathbb{N},<)}$-**invariant** on M^k, i.e., if for $(x_i : i < k), (y_i : i < k) \in M^k$, $(x_0, \ldots, x_{k-1}) \sim_{(\mathbb{N},<)} (y_0, \ldots, y_{k-1})$ implies $R(x_0, \ldots, x_{k-1}) \Leftrightarrow R(y_0, \ldots, y_{k-1})$ where we put*

$$(x_i : i < k) \sim_{(\mathbb{N},<)} (y_i : i < k)$$

if of all $i, j < k$:
$x_i < x_j \Leftrightarrow y_i < y_j$,
$x_i = x_j \Leftrightarrow y_i = y_j$,
$x_i > x_j \Leftrightarrow y_i > y_j$.

Recognizing canonical relations

Recognizing canonical relations

Proposition

There is exactly eight canonical binary relations on $\mathbb{N}$ *:*

$$\top, \bot, =, \neq, <, >, \leqslant, \geqslant .$$

$\top$ *and* $=$ *are the only* **canonical equivalence relations** *on* $\mathbb{N}$*.*

Canonical equivalence relations

Theorem (Erdös-Rado 1950)

There is exactly 2^k *canonical equivalence relations on* $\mathbb{N}^{[k]}$:

$$(x_i : i < k) \sim_I (y_i : i < k) \Leftrightarrow (x_i : i \in I) = (y_i : i \in I),$$

for $I \subseteq \{0, ..., k-1\}$, *i.e., for every equivalence relation* E *on*

$$\mathbb{N}^{[k]} = \{(x_i : i < k) \in \mathbb{N}^k : x_0 < x_1 < \cdots < x_{k-1}\}$$

there is an infinite set $M \subseteq \mathbb{N}$ *and a set* $I \subseteq \{0, ..., k-1\}$ *such that*

$$E|M^{[k]} = \ \sim_I |M^{[k]}.$$

Tukey meets Ramsey

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Definition

A collection $\mathcal{F}$ of finite subsets of $\mathbb{N}$ is a **barrier** *if every infinite subset of $\mathbb{N}$ has an initial segment in $\mathcal{F}$ and if no two distinct elements of $\mathcal{F}$ are comparable under inclusion.*

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Theorem (Pudlak-Rödl, 1982)

For every equivalence relation E on some **barrier** *$\mathcal{B}$ on $\mathbb{N}$ there is an infinite set $M \subseteq \mathbb{N}$ and an* **internal irreducible** *mapping f on $\mathcal{B}|M$ such that $E \restriction (\mathcal{B}|M) = E_f$.*

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Theorem (T., 2012)

Let $\mathcal{V}$ be a **selective ultrafilter** *on $\mathbb{N}$ and let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ such that $\mathcal{U} \leq_T \mathcal{V}$. Then $\mathcal{U}$ is Rudin-Keisler isomorphic to a countable Fubini power of $\mathcal{V}$.*

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Corollary

Selective ultrafilters are **Tukey minimal** *members of $\beta\mathbb{N} \setminus \mathbb{N}$.*

Ramsey basis results for $\mathbb{Q}$

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Theorem (Laver 1970)

For every natural number k and every set $R \subseteq \mathbb{Q}^k$ there is $M \subseteq \mathbb{Q}$ **order-isomorphic to** $\mathbb{Q}$ *such that*
*R is a $(\mathbb{Q}, <, <')$-***canonical relation** *on M,*
where $<$ is the usual ordering on $\mathbb{Q}$ and where $<'$ is a well-order of $\mathbb{Q}$ of order-type ω.

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Theorem (Devlin 1979)

Among canonical equivalence relations E on $\mathbb{Q}^{[k]}$ with **finitely many classes** *there is* **the finest canonical equivalence relation** *that has exactly $t_k = T_{2k+1}$ classes,*
where T_n are **tangent numbers** *given by*
$\tan z = \sum_{n=0}^{\infty} \frac{T_n}{n!} z^n$.

Thus, $t_1 = 1,\ t_2 = 2,\ t_3 = 16,\ t_4 = 272,\ \ldots .$

Remark

- *There are also results about arbitrary canonical equivalence relations on* $\mathbb{Q}^{[k]}$ *(Vuksanovic 2012) but perhaps here we could have a result that would match Devlin's in its clarity.*

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Question

Let $\mathbb{A}$ be a countable ultrahomogeneous structure.

- *Under which condition on $\mathbb{A}$ we can find an expansion $\mathbb{A}^*$ with finitely many relations such that every subset R of some finite power $\mathbb{A}^k$ is $\mathbb{A}^*$-***canonical*** *when restricted to some substructure of $\mathbb{A}$ isomorphic to $\mathbb{A}$?*

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- *Under which additional assumptions (if any) can we conclude that on any finite symmetric power $\mathbb{A}^{[k]}$ there is* **the finest** *canonical equivalence relation with finitely many classes?*

Ramsey basis results for $\mathrm{top}(\mathbb{Q})$

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Theorem (T., 1994)

There is an equivalence relation E_{osc} on $\mathbb{Q}^{[2]}$ with infinitely many classes $e_1, e_2, ..., e_k, ...$ such that if for some positive integer k the closure $\overline{X}$ of some subset X of $\mathbb{Q}$ has its kth Cantor-Bendixson derivative nonempty then

$$X^{[2]} \cap e_i \neq \emptyset \text{ for all } 2 \leq i \leq 2k.$$

Moreover, if X is not a discrete subspace of $\mathbb{Q}$ then $X^{[2]} \cap e_1 \neq \emptyset$.

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The class of equivalence relations on $\mathbb{Q}^{[2]}$ (even those with finitely many equivalence classes) does not have finite Ramsey basis.

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Theorem (T., 1994)

The class of equivalence relations on $\mathbb{Q}^{[2]}$ with **open equivalence classes** *has 26-element Ramsey basis.*

Basis problems for $\mathbb{R}$

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Theorem (Sierpinski, 1933)

Let $<$ be the usual lexicographic ordering of $2^{\mathbb{N}}$, let $<'$ be a well-ordering of $2^{\mathbb{N}}$ and let $\mathbb{S}$ denote the expanded structure $(2^{\mathbb{N}}, \Delta, <, <')$. Then for every positive integer k the finest $\mathbb{S}$-canonical equivalence relation $\sim_{\mathbb{S}}^{k}$ on $(2^{\mathbb{N}})^{[k]}$ that has $k!(k-1)!$ many classes has the property that every **uncountable** *$X \subseteq 2^{\mathbb{N}}$ realizes all the classes.*

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Conjecture (Galvin, 1970)

Every equivalence relation on $\mathbb{R}^{[2]}$ with finitely many classes is $\mathbb{S}$-canonical when restricted to some **topological copy of** $\mathbb{Q}$.

Basis problems for ω_1, ω_2, ...

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For every positive integer k there is an equivalence relation E on $[\omega_k]^{k+1}$ with uncountably many classes such that every uncountable subset X of ω_k realizes all the classes of E.

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Galvin's Conjecture implies $2^{\aleph_0} > \aleph_\omega$.

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Question

Does Galvin's Conjecture require the continuum to be above the first weakly inaccessible cardinal?

Borel restriction

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The Cantor space $2^{\mathbb{N}}$ as the Borel structure:

$$(2^{\mathbb{N}}, <, \Delta)$$

where $<$ is the lexicographical ordering and Δ the distance function:

$$\Delta(x, y) = \min\{n : x(n) \neq y(n)\}.$$

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Theorem (Galvin, 1968)

*For every positive integer k every Borel set $R \subseteq (2^{\mathbb{N}})^k$ is $(2^{\mathbb{N}}, <, \Delta)$-**canonical** on some perfect set $P \subseteq 2^{\mathbb{N}}$.*

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Theorem (Galvin 1968, Blass 1981)

Among $(2^{\mathbb{N}}, <, \Delta)$-canonical Borel equivalence relation on a given finite symmetric power $[2^{\mathbb{N}}]^k$ with **finitely many classes** *there is the finest one which has exactly $(k-1)!$ many classes.*

Theorem (Taylor 1979, Lefmann 1983, Vlitas 2014)

- *There is exactly* **two** $(2^{\mathbb{N}}, <, \Delta)$*-canonical Borel equivalence relations on* $[2^{\mathbb{N}}]^2$ *with* **countably many classes**: $\top$ *and* E_Δ.

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  1. $x_0 = x_0$,
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  3. $x_1 = y_1$,
  4. $x_0 = y_0$ *and* $x_1 = y_1$,
  5. $\Delta(x_0, x_1) = \Delta(y_0, y_1)$ *and* $x_0 = x_0$,
  6. $\Delta(x_0, x_1) = \Delta(y_0, y_1)$ *and* $x_0 = y_0$,
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  7. $\Delta(x_0, x_1) = \Delta(y_0, y_1)$ *and* $x_1 = y_1$.
- *There is exactly* **twenty five** $(2^{\mathbb{N}}, <, \Delta)$*-canonical Borel equivalence relations on* $[2^{\mathbb{N}}]^3$

The arrow-notation

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For two structures $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same type, set

$$\binom{\boldsymbol{B}}{\boldsymbol{A}} = \{\boldsymbol{A}' : \boldsymbol{A}' \text{ is a substructure of } \boldsymbol{B} \text{ isomorphic to } \boldsymbol{A}\}.$$

For $\boldsymbol{A}$,$\boldsymbol{B}$ and $\boldsymbol{C}$ of the same type and cardinals λ and τ, let

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_{\lambda,\tau}$$

denote the statement that for every coloring $\chi : \binom{\boldsymbol{C}}{\boldsymbol{A}} \to \lambda$

there is $\boldsymbol{B}' \in \binom{\boldsymbol{C}}{\boldsymbol{B}}$ such that χ on $\binom{\boldsymbol{B}'}{\boldsymbol{A}}$ has $\leq \tau$ values.

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$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_{\lambda} \quad \text{iff} \quad \boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_{\lambda,1},$$

$$\boldsymbol{C} \to [\boldsymbol{B}]^{\boldsymbol{A}}_{\lambda} \quad \text{iff} \quad \boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_{\lambda,\lambda-1}.$$

Examples

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Theorem (Galvin 1970)

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Theorem (Devlin, 1979)

Fix a positive integer k *and let* $t_k = \tan^{(2k-1)}(0)$ *and consider the rationals* $\mathbb{Q}$ *as ordered set.*

- $\mathbb{Q} \to (\mathbb{Q})^k_{l,t_k}$ *for all* $l < \omega$.
- $\mathbb{Q} \not\to [\mathbb{Q}]^k_{t_k}$.

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Conjecture (Galvin 1970)

For every positive integer k,

- $2^{\aleph_0} \to (\aleph_1)^k_{l,k!(k-1)!}$ *for all* $l < \omega$,
- $2^{\aleph_0} \not\to [\aleph_1]^k_{k!(k-1)!}$.

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$\aleph_k \not\to [\aleph_1]^{k+1}_{\aleph_0}$ *for all positive integers* k.

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$\aleph_k \not\to [\aleph_1]^{k+1}_{\aleph_0}$ *for all positive integers* k.

Theorem (Devlin 1979, Folklore)

Fix a positive integer k *and let* $t_k = \tan^{(2k-1)}(0)$. *Let* $\mathcal{R}$ *denote the random graph and let* K_k *denote the complete graph on* k *vertices.*

- $\mathcal{R} \to (\mathcal{R})^{K_k}_{l,t_k}$ *for all* $l < \omega$.
- $\mathcal{R} \not\to [\mathcal{R}]^{K_k}_{t_k}$.

Ramsey degrees in Fraïssé classes

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Let $\mathcal{K}$ be a given class of finite L-structures.
For $\boldsymbol{A} \in \mathcal{K}$, let $t(\boldsymbol{A}, \mathcal{K})$ be the **minimal number** *t (if it exists) such that for every $\boldsymbol{B}$ in $\mathcal{K}$ and $l < \omega$ there exists $\boldsymbol{C}$ in $\mathcal{K}$ such that*

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_{l,t}.$$

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For $\boldsymbol{A} \in \mathcal{K}$, let $t(\boldsymbol{A}, \mathcal{K})$ be the **minimal number** *t (if it exists) such that for every $\boldsymbol{B}$ in $\mathcal{K}$ and $l < \omega$ there exists $\boldsymbol{C}$ in $\mathcal{K}$ such that*

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If $\mathrm{Age}(\boldsymbol{F}^*)$ *has the* **Ramsey property** *, then*

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and so, in particular $t(\boldsymbol{A}, \mathrm{Age}(\boldsymbol{F})) < \infty$ *for all* $\boldsymbol{A} \in \mathrm{Age}(\boldsymbol{F})$.

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For $\boldsymbol{F}$ *and* $\boldsymbol{F}^*$ *as above, we say that* $\boldsymbol{F}^*$ *has the* **expansion property** *relative to* $\boldsymbol{F}$ *whenever*

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If the expansion $\boldsymbol{F}^*$ *has both the* **Ramsey property** *and the* **expansion property** *relative to* $\boldsymbol{F}$, *then*

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Theorem (Zucker 2014)

Let $\boldsymbol{F}$ be a countable locally finite ultrahomogeneous structure. If the group $\mathrm{Aut}(\boldsymbol{F})$ has **metrizable universal minimal flow** *then $t(\boldsymbol{A}, \mathrm{Age}(\boldsymbol{F})) < \infty$ for all $\boldsymbol{A} \in \mathrm{Age}(\boldsymbol{F})$.*