Ordinal definability in extender models

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Outline

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- We assume ZFC consistent.
- ZFC leaves many natural questions undecided.
- Example: Are all projective sets of reals Lebesgue measurable? ("Yes" for analytic sets.)
- Projective sets $P \subseteq \mathbb{R}$ have form:

$$P(x) \iff \exists x_1 \forall x_2 \exists x_3 \ldots Qx_n[\varphi(x, x_0, \ldots, x_n)]$$

  where φ is closed (n odd) or open (n even) and quantifiers range over reals.
- P above is $\underset{\sim}{\Sigma}^1_n$.

- Some undecided questions can be decided by *large cardinal* axioms, which are strengthenings of the Axiom of Infinity.
- V is arranged in the Von Neumann hierarchy:
  - $V_0 = \emptyset$,
  - $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,
  - $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for limit ordinals λ.

- An *inaccessible cardinal* is a regular, strong limit cardinal κ: For all $\alpha < \kappa$ we have:

$$2^\alpha < \kappa,$$

$$\forall f[f : \alpha \to \kappa \implies \sup(\mathrm{range}(f)) < \kappa].$$

- If κ is inaccessible then $V_\kappa \models \mathrm{ZFC}$, and κ is a limit of α such that $V_\alpha \models \mathrm{ZFC}$.
- ZFC does not prove the existence of inaccessibles.

- A class $M \subseteq V$ is *transitive* if

$$x \in M \implies x \subseteq M.$$

- Given a transitive class M, an *elementary embedding* $j : V \to M$ is a class function such that

$$V \models \varphi(x) \iff M \models \varphi(j(x))$$

  for all sets x and formulas φ.
- We have $j(\kappa) \geq \kappa$ for all ordinals κ.
- There is an ordinal κ with $j(\kappa) > \kappa$; the least is the *critical point* of j, denoted $\mathrm{crit}(j)$.

- We say that κ is *measurable* iff $\kappa = \mathrm{crit}(j)$ for some $j : V \to M$.
- We can then define the *derived normal measure* U, a countably complete ultrafilter over κ, by

$$X \in U \iff \kappa \in j(X).$$

- If κ is measurable then κ is inaccessible, a limit of inaccessibles, a limit of limits of inaccessibles, etc.

- ZFC proves that all $\undertilde{\Sigma}^1_1$ sets are Lebesgue measurable, but $\undertilde{\Sigma}^1_2$ undecided.
- If there is a measurable cardinal then all $\undertilde{\Sigma}^1_2$ sets are Lebesgue measurable.
- However, it is consistent to have a measurable cardinal together with a Σ^1_3 good wellorder of the reals.
- Stronger large cardinals decide more.
- *Inner model theory* focuses on the study of canonical models of set theory with large cardinals.

Outline

- First canonical inner model is L, Gödel's constructible universe:
- $L = \bigcup_\alpha L_\alpha$ where $\langle L_\alpha \rangle$ is the increasing hierarchy of sets:
  - $L_0 = \emptyset$,
  - $L_{\alpha+1} = \mathcal{P}_{\text{def}}(L_\alpha)$,
  - L_λ is union for limit λ.
- Here $\mathcal{P}_{\text{def}}(M)$ is the set of all $X \subseteq M$ such that X is definable over M from parameters in M.
- Restricted version of the V_α hierarchy: have $L_\alpha \subseteq V_\alpha$.
- $L_{\alpha+1}$ has the same cardinality as L_α (α infinite).
- $L_{\omega+1}$ is countable, while $V_{\omega+1}$ is uncountable.

Theorem (Gödel)

L satisfies ZFC + GCH+*"$V = L$".*

- L is well understood, through *condensation* and *fine structure*:
- Condensation: for any $X \preccurlyeq_1 L_\alpha$, there is $\beta \leq \alpha$ such that $X \cong L_\beta$.
- This leads to GCH.
- L has a Σ^1_2 wellorder of the reals.
- But large cardinals very limited in L...

- If κ is inaccessible, then $L \models$"κ is inaccessible".
- But L does *not* satisfy "There is a measurable cardinal".
- Inner model theory is focused on construction and analysis of inner models generalizing L, but having large cardinals.

Outline

- Another universe introduced by Gödel was HOD, the *hereditarily ordinal definable* sets.

Definition

(Gödel) x is OD or *ordinal definable* iff there is an ordinal α and a formula φ such that

$$x \text{ is the unique set } x' \text{ such that } \varphi(x', \alpha).$$

- Ordinals, pairs of ordinals, are OD.
- $\forall x \in L[x \in \mathrm{OD}]$.
- "OD" has a first-order reformulation (modulo ZF).

Definition

(Gödel) A set x is *hereditarily ordinal definable* iff

- $x \in \mathrm{OD}$ and
- $\forall y \in x[y \in \mathrm{OD}]$ and
- $\forall y \in x[\forall z \in y[z \in \mathrm{OD}]]$ and
- ...

HOD denotes the class of all such sets.

- $L \subseteq \mathrm{HOD}$.
- ZF proves that $\mathrm{HOD} \models \mathrm{ZFC}$.
- AC because we can wellorder the definitions from ordinals.
- HOD need not satisfy "$V = \mathrm{HOD}$".
- In contrast to L, which satisfies "$V = L$".

- Every real which is definable over the reals without parameters, is in HOD.
- If $\mathbb{R} \subseteq \mathrm{HOD}$ then there is a definable wellorder of the reals.
- If measurable cardinals exist then $L \subsetneq \mathrm{HOD}$, in fact that $\mathrm{HOD} \models$"$\mathbb{R} \cap L$ is countable".

- Elements of HOD are in some sense canonical, but not in the absolute way true of constructible sets.
- Question: Does HOD satisfy the GCH? Does HOD have condensation properties like L?
- The answers are not decided.

Theorem (Roguski)

For any countable transitive model M of ZFC there is a larger model N of ZFC such that $M = \mathrm{HOD}^N$.

Outline

- Deeply connected to large cardinal axioms are *determinacy* axioms.
- The *Axiom of Determinacy*, AD, asserts that every two player game of perfect information, of length ω, with integer moves, is determined.
- Fix a set $A \subseteq \omega^\omega$. That is, A is a set of functions $x : \omega \to \omega$; here $\omega = \aleph_0 = \mathbb{N}$ is the set of natural numbers.
- We define a game $\mathcal{G}_A$ associated to A.

- The game $\mathcal{G}_A$:
- Two players, I and II.
- I first plays $x_0 \in \omega$,
- then II plays $x_1 \in \omega$,
- then I plays $x_2 \in \omega$,
- then II plays $x_3 \in \omega$,
- ...and so on..., through ω-many rounds.
- This produces a sequence $x = \langle x_n \rangle_{n<\omega}$, a *run* of the game.
- We say that I *wins* the run iff $x \in A$.
- We say that $\mathcal{G}_A$ (or just A) is *determined* iff there is an (always) winning strategy for one of the players.

- Determinacy of analytic games undecided by ZFC. But:

Theorem (Martin)

Borel games A are determined. If there is a measurable cardinal, then all analytic games are determined.

- Combined with other results, this gives that all $\utilde{\Sigma}^1_2$ sets are Lebesgue measurable, given a measurable cardinal.

- AD contradicts AC.
- Can consider restrictions of AD to simpler sets of reals.
- $L(\mathbb{R})$ given by constructing above $\mathbb{R}$.
- $L(\mathbb{R}) =$ smallest transitive proper class satisfying ZF, containing all reals and ordinals.
- *Strong cardinals* are a strengthening of measurable cardinals.
- A *Woodin cardinal* is an even stronger large cardinal property. If δ is Woodin then δ is an inaccessible limit of κ such that $V_\delta \models$"κ is a strong cardinal", and more.

Theorem (Martin, Steel, Woodin)

If there are ω many Woodin cardinals and a measurable above their supremum, then $L(\mathbb{R}) \models \mathrm{AD}$.

Theorem (Woodin)

$\mathrm{ZF} + \mathrm{AD}$ *is equiconsistent with* $\mathrm{ZF}+$*"There are infinitely many Woodin cardinals".*

Theorem (Woodin, building on work of Steel)

Suppose $L(\mathbb{R}) \models \mathrm{AD}$. Then $\mathrm{HOD}^{L(\mathbb{R})}$ is a hybrid strategy premouse, $\mathrm{HOD}^{L(\mathbb{R})}$ can be analysed in detail, has condensation properties, and satisfies GCH.

Outline

- Inner model theory is focused on the construction and analysis of *extender models* $L[\vec{E}]$.
- $L[\vec{E}]$ is constructed in a hierarchy $\langle M_\alpha \rangle_{\alpha \leq \lambda}$ much like L, but we also have a predicate $\vec{E}$ encoding extra information.
- The M_α are also extender models, and also called *(pre)mice*.
- $\vec{E}$ is a sequence of (short) *extenders*.

- An extender F is a system of ultrafilters, coding a partial elementary embedding.
- The extenders F appearing in the sequence $\vec{E}$ define elementary embeddings over (some fragment of) M.
- Given a mouse M and an extender F over M, we can form the *ultrapower* of M by F, denoted $\mathrm{Ult}(M, F)$. We also define a natural elementary embedding

$$i_F^M : M \to \mathrm{Ult}(M, F),$$

  the *ultrapower embedding*.
- In some cases F is essentially just an ultrafilter, and then $\mathrm{Ult}(M, F)$ is the usual model theoretic ultrapower.
- If $M \models \mathrm{ZFC}$ then i_F^M is fully elementary.

- The first mouse beyond L is $0^{\#}$.
- It has universe some L_λ.
- It has only one extender E in its sequence $\vec{E}$.
- E is equivalent to a single ultrafilter over L_λ.
- $0^{\#} = (L_\lambda, E)$
- A key property of $0^{\#}$ is that $N = \mathrm{Ult}(L_\lambda, E)$ is wellfounded.
- $N = L_\gamma$ for some $\gamma > \lambda$.

- We can make sense of $F = i_E(E)$, which is then an ultrafilter over L_β, and define

$$\mathrm{Ult}(0^\#, E) = (L_\beta, F).$$

- Let $M_{(0)} = 0^\#$ and $M_{(1)} = \mathrm{Ult}(0^\#, E)$. We can go on to define

$$M_{(\alpha+1)} = \mathrm{Ult}(M_{(\alpha)}, E_\alpha),$$

where $M_{(\alpha)} = (L_{\gamma_\alpha}, E_\alpha)$, and take direct limits at limit stages.

- Key property of $0^\#$: *every $M_{(\alpha)}$ has wellfounded universe.*

- This wellfoundedness requirement is called *iterability*.
- Iterability is not first-order.
- A structure $M = (L_\lambda, E)$ as above is *sound* iff every $x \in M$ is Σ_1-definable over M without parameters.
- $0^\#$ is the *unique* such sound iterable structure which satisfies some further first-order requirements.
- Uniqueness proved by *comparison*.
- Given two candidates M, N, form iterations $\langle M_{(\alpha)} \rangle_{\alpha \in \mathrm{OR}}$ and $\langle N_{(\alpha)} \rangle_{\alpha \in \mathrm{OR}}$ of M, N.
- Show that there are α, β such that $M_{(\alpha)} = N_{(\beta)}$.
- Use soundness to deduce that $M = N$.

- Beyond $0^\#$, extender models can have many different extenders E in their sequence $\vec{E}$.
- We need to be able to choose arbitrary extenders and form ultrapowers, always producing wellfounded models.
- Woodin cardinals introduce new complexities to extender models.
- The theory of extender models at the level of Woodin cardinals was developed by Martin, Steel and Mitchell. It required the introduction of *iteration trees*.

- In the example of iterating $0^\#$ above, the iteration was *linear*, meaning that at stage α, we used the extender E_α to form an ultrapower of $M_{(\alpha)}$.
- In an iteration tree, the α^{th} extender E_α used in the tree may be applied to a model $M_{(\beta)}$ for some $\beta \leq \alpha$, forming

$$M_{(\alpha+1)} = \mathrm{Ult}(M_{(\beta)}, E).$$

- This leads to a *tree* of models, with elementary embeddings along the branches.
- Problem at limit stages: Must *choose a cofinal branch* through the tree, and form the direct limit of the models along that branch.

- The existence of branches and method of choosing good branches is a deep problem.
- An *iteration strategy* for a premouse M is a function which chooses branches, always ensuring wellfoundedness.
- M is *iterable* if an iteration strategy for M exists.
- Then, roughly, we say that M is a *mouse*.

- Theorem of Woodin stated earlier:
- Assume AD holds in $L(\mathbb{R})$. Then

$$\mathrm{HOD}^{L(\mathbb{R})} = M[\Sigma],$$

where M is a proper class premouse with ω many Woodin cardinals, and Σ is a partial iteration strategy for M.

- So our understanding of extender models yields much information about $\mathrm{HOD}^{L(\mathbb{R})}$ under AD, e.g. GCH.

Outline

- What is $\mathrm{HOD}^{L[\vec{E}]}$, the HOD as computed in a mouse $L[\vec{E}]$?
- The model $L[U]$ for one measurable cardinal κ is (similar to) a premouse, with extender sequence consisting of a single normal measure U. Built in a hierarchy like L.
- U, κ are such that $L[U] \models$"κ is measurable and U is the associated normal measure".
- Kunen showed that $L[U] \models$"U is the unique normal measure".
- It follows that $L[U] \models$"$V = \mathrm{HOD}$" (like for L).

- Given set or class X of ordinals, HOD_X is defined like HOD, but we allow definitions from X and ordinal parameters.
- $L[\vec{E}]$ satisfies "$V = \mathrm{HOD}_{\vec{E}}$".
- Not obvious that M satisfies "$V = \mathrm{HOD}$", as $\vec{E}$ might not be definable over (the universe of) M.
- In the case of $L[U]$ the uniqueness of D made it work.

- For $n \leq \omega$, M_n denotes the minimal proper class extender model with n Woodin cardinals.
- (Steel) For each $n \leq \omega$, $\vec{E}^{M_n}$ is definable over the universe of M_n, so M_n satisfies "$V = \mathrm{HOD}$".
- This is more subtle than the $L[U]$ case, particularly because of the fact that Woodin cardinals lead to non-linear iterations.
- Even though M_n is iterable, M_n does not satisfy "I am iterable".
- However, it does know a significant portion of its own iteration strategy, which is important in Steel's proof.

- Steel's result generalizes to:
- Theorem: Let M be a mouse satisfying ZFC, such that M satisfies "I am sufficiently iterable". Then:
  1. (Woodin) M satisfies "$V = \mathrm{HOD}$",
  2. (S.) $\vec{E}^M$ is definable over the universe of M.
- The dependence on self-iterability in the theorem is a strong limitation.
- There are examples of proper class mice M which satisfy "$\mathbb{R} \not\subseteq \mathrm{HOD}$".

- Steel asked: Let M be a mouse satisfying ZFC. Does M satisfy "$V = \mathrm{HOD}_X$ for some $X \subseteq \omega_1$"?
- It turns out the answer is "yes", even without any self-iterability assumptions:

Theorem (S.)

Let M be a mouse satisfying ZFC. Then $\vec{E}^M$ is definable over the universe of M from the parameter $X = \vec{E}^M \restriction \omega_1^M$. Therefore, M satisfies "$V = \mathrm{HOD}_X$".

- Can the "ω_1" be reduced? A recent partial result:

Theorem (S.)

Let M be any tame mouse satisfying ZFC. *Then $\vec{E}^M$ is definable over the universe of M from some $x \in \mathbb{R} \cap M$. Moreover, M satisfies "There is a $\Sigma_2^{\mathcal{H}_{\omega_2}}(x)$ wellorder of the reals, for some $x \in \mathbb{R}$".*

- A *tame* mouse has no $E \in \vec{E}$ *overlapping a Woodin cardinal*.
- Steel and Schindler showed that every tame mouse satisfying ZFC knows a significant piece of its own iteration strategy.
- Their results are important in the proof of the theorem. But their methods break down for non-tame mice.

- Question: Let M be a mouse satisfying ZFC, and suppose that $\mathrm{HOD}^M \subsetneq M$. What is the structure of HOD^M?
- At present the picture here is not very well understood, even in the simplest cases.
- A full solution would probably relate to the analysis of the HOD of determinacy models, such as $\mathrm{HOD}^{L(\mathbb{R})}$.

- Assuming determinacy, for sufficiently complex reals x, $\mathrm{HOD}^{L[x]}$ has some properties analogous to $\mathrm{HOD}^{L(\mathbb{R})}$.
- $L[x]$ satisfies "$V_{\omega_1}^{\mathrm{HOD}}$ is a mouse".
- Not known whether $L[x] \models$"$V_{\omega_2}^{\mathrm{HOD}}$ is a (pre)mouse".
- Woodin has several partial results in this direction.

- Some mice have universe of the form $L[x]$ for a real x of high complexity.
- We would probably have to solve the $\mathrm{HOD}^{L[x]}$ problem.
- The part of $\mathrm{HOD}^{L[\vec{E}]}$ which is difficult to analyze is below $\omega_2^{L[\vec{E}]}$ (or $\omega_3^{L[\vec{E}]}$). Above that point, there are positive results.

Theorem (S.)

Let M be an iterable tame mouse satisfying ZFC*. Suppose that* $H = \mathrm{HOD}^M \subsetneq M$*. Then:*

- $H = L[\vec{E}^H, t]$ *is a mouse over a set* $t \subseteq \omega_2^M$*,*
- *M is a generic extension of H,*
- $M = H[\mathbb{e}]$ *where* $\mathbb{e} = \vec{E}^M \restriction \omega_2^M$*,*
- $\vec{E}^M \restriction [\omega_2^M, \mathrm{OR}^M)$ *is given by lifting* $\vec{E}^H$ *to the generic extension.*

The set t is just

$$t = \mathrm{Th}_{\Sigma_3}^{(\mathcal{H}_{\omega_2})^M}(\omega_2^M).$$

Theorem (S.)

Let M be an iterable mouse satisfying ZFC*, below a Woodin limit of Woodins. Let $\delta = \omega_2^M$, let $H = \mathrm{HOD}^M$ and t be as above. Then there is a premouse W such that:*

- *W satisfies "δ is Woodin" and t is generic over W,*
- *$H = W[t]$,*
- *$M = H[\mathbb{e}]$ where $\mathbb{e} = \vec{E}^M \upharpoonright \delta$, and*
- *$\vec{E}^M \upharpoonright [\delta, \mathrm{OR}^M)$ is determined by "translating" $\vec{E}^W$ above δ.*

Conjecture: Let M be any iterable mouse satisfying ZFC. Let $\delta = \omega_3^M$. Then the conclusion of the preceding theorem holds. (Maybe also for ω_2^M?)

- Question: What is the full structure of $\mathrm{HOD}^{L[\vec{E}]}$? Is there an analysis analogous to that for $\mathrm{HOD}^{L(\mathbb{R})}$?
- Question: Let M be a non-tame mouse satisfying ZFC. Does M satisfy "$V = \mathrm{HOD}_x$ for some real x"?