Developments in unstable theories

Itay Kaplan, HUJI

02/08/2016, Logic Colloquium 2016, Leeds, England

Some ancient history

Suppose we wish to study some properties of first order theories.
For example, let $I(T, \lambda)$, for a first order theory T and an infinite cardinal λ, denote the number of models $M \models T$ of size λ up to isomorphism.

Theorem

(Morley 1965) For a countable theory T, $I(T, \aleph_1) = 1$ iff $I(T, \lambda) = 1$ for any uncountable cardinal λ.

Why Classify?

Morley studied the behavior of $I(T, \lambda)$ when it is the smallest possible value: 1.

What happens if one considers the largest possible value (which is 2^λ)?

Theorem

(Shelah's Main Gap, 1982) Let T be a countable complete first order theory. Then one of the following possibilities happen:

- *$I(T, \lambda) = 2^\lambda$ for every uncountable cardinal λ.*
- *For every $\alpha > 0$, $I(T, \aleph_\alpha) < \beth_{\omega_1}(|\alpha|)$.*

The function $I(T, \lambda)$ was later completely classified by Bradd Hart, Ehud Hrushovski and Michael C. Laskowski.

Stability

It turns out that the dichotomy in the main gap depends on combinatorial properties of the theory. If T is unstable, for instance, then $I(T, \lambda) = 2^{\lambda}$ always.

Definition

T is *stable* if it does not have the order property: there is no formula $\varphi(x, y)$ and tuples $\langle a_i, b_j \mid i, j < \omega \rangle$ (in the monster model $\mathfrak{C}$) such that $\mathfrak{C} \models \varphi(a_i, b_j)$ iff $i < j$.
+ many other equivalent definitions.

Examples

Equivalence relations, algebraically closed fields, abelian groups, the free group (by work of Sela).

Let's take a look at another natural invariant of a theory.

Saturation

Recall:

Definition

Recall that a structure M is κ-saturated if for every $A \subseteq M$ of size $< \kappa$ and any complete type $p(x) \in S(A)$, p is realized in M.

Exact saturation

Suppose that T is a first order theory.

Say that T has *exact saturation at a cardinal* κ if there is a model $M \models T$ such that M is κ-saturated but not κ^+-saturated.

There are a couple of natural questions regarding this notion:

1. Given a theory T, what are the cardinals κ for which T has exact saturation at κ?

2. For a cardinal κ, can we characterize the theories which have exact saturation in κ?

Exact saturation in stable theories

Let us start with stable theories, where we can answer both these questions with roughly the same answer.

Theorem

(Shelah) Assume T is stable. Then for all $\kappa > |T|$, T has exact saturation at κ.

Idea of proof: let I be an indiscernible set of cardinality κ.
Call a type $p \in S^{<\omega}(I)$ a *D-type* if there is some $I_0 \subseteq I$ of cardinality $\leq |T|$ such that $p|_{I_0} \models p$.
Stability implies that there is a κ-saturated D-model M (every finite tuple $a \in M^n$ satisfies $\mathrm{tp}(a/M) \in D$).
But then M is not κ^+-saturated: the type of a new element is not realized.

Non-stable theories

Fact

If T is not stable then T has exact saturation at any regular κ.

Proof.

(T countable) Let $M = \bigcup_{i<\kappa} M_i$, $|M_0| = \aleph_0$, $|M_{i+1}| = 2^{|M_i|}$, M_{i+1} is $|M_i|$-saturated.

By instability, $|S(M)| > \beth_\kappa$ and the number of types over M invariant over M_i is $\leq 2^{2^{|M_i|}} < \beth_\kappa$, there is $p \in S(M)$ which splits over every M_i.

I.e., for each $i < \kappa$, there is $\varphi(x, a) \wedge \neg\varphi(x, b) \in p$ where $a \equiv_{M_i} b$.

Let q be the union of these formulas. Then q is not realized in M. □

Non-stable theories

Question: What about singular cardinals?
Our expectation is that having or not having exact saturation at a singular cardinal λ should not depend on λ in any deep way but just on properties of the theory.
In order to explain the next result, I'll need to introduce a new class.

Simple theories

Definition

We say that $\varphi(x, y)$ has the tree property if there is some number k and tuples $a_s \in \mathfrak{C}^y$ for $s \in \omega^{<\omega}$ such that for every path $\eta : \omega \to \omega$, $\{\varphi(x, a_{\eta \restriction n}) \mid n < \omega\}$ is consistent and for every $s \in \omega^{<\omega}$ the set $\{\varphi(x, a_{s \frown i}) \mid i < \omega\}$ is k-inconsistent (every k-element subset is inconsistent).

Definition

A theory T is called *simple* if no formula has the tree property.

Examples

Any stable theory, The random graph, ACFA (algebraically closed field with a generic automorphism).

Exact saturation in simple theories

Definition

(*Jensen's Square principle*) Let κ be an uncountable cardinal; $\square_\kappa$ (square-κ) is the following condition:
There exists a sequence $\langle C_\alpha \mid \alpha \in \mathrm{Lim}\,(\kappa^+) \rangle$ such that:

1. C_α is a closed unbounded subset of α.
2. If $\beta \in \mathrm{Lim}\,(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$.
3. If $\mathrm{cof}\,(\alpha) < \kappa$, then $|C_\alpha| < \kappa$.

Fact

By work of Jensen, $\square_\kappa$ holds in L for every uncountable cardinal κ. In addition it is relatively easy to achieve $\square_\kappa$ by forcing.

Theorem (w/ Saharon Shelah and Pierre Simon)

Suppose that T is simple, μ is singular with $|T| < \kappa = \mathrm{cof}\,(\mu)$, $\mu^+ = 2^\mu$ and $\square_\mu$ holds. Then T has exact saturation at μ.

So simple theories are a lot like stable theories.

Dense linear order

Example

DLO (the theory of $(\mathbb{Q}, <)$ — a dense linear order without end points) has exact saturation only at regular cardinals. Say that DLO has *singular compactness*.

Proof.

Suppose $(M, <) \models DLO$ is saturated at κ where κ is singular.

Let $p \in S_1(A)$ for $|A| = \kappa$. Then p is a cut in A, say (I_1, I_2).

But then the cofinalities of I_1, I_2 must be smaller than κ, i.e., there are $(J_1, J_2) \subseteq (I_1, I_2)$ cofinal with $|J_1|, |J_2| < \kappa$.

Then $p|_{J_1 J_2} \models p$ and is realized. $\qquad\square$

NIP theories

Definition

A theory is NIP (No Independence Property; also, dependent) if for no formula $\varphi(x, y)$ are there $\langle a_i \mid i < \omega \rangle$ and $\langle b_s \mid s \subseteq \omega \rangle$ such that $\varphi(a_i, b_s)$ iff $i \in s$.

Example

Any stable theory, DLO, RCF, $\mathbb{Q}_p$.

Fact

(Shelah) A simple, NIP theory is stable.

Hence we see that there are NIP theories (DLO) that have singular compactness, unlike the stable and simple cases.

Question: what are the NIP theories that have singular compactness?

Distal theories

Definition

A theory T is called *distal* if whenever $I_1 + a + I_2$ is an indiscernible sequence, with I_1, I_2 infinite and $I_1 + I_2$ is A-indiscernible then $I_1 + a + I_2$ is A-indiscernible.

Another, equivalent definition is: Suppose I is dense indiscernible. Then any distinct cuts c_1, c_2 in I have weakly orthogonal limit types.
Note that this definition, as opposed to that of stability and NIP is not of the form "some combinatorial pattern does not occure", and indeed it is not invariant under reducts.

Example

DLO, RCF, $\mathbb{Q}_p$. However, no stable theory is distal (any indiscernible sequence is an indiscernible set).

Distal theories

Theorem (w/ Saharon Shelah and Pierre Simon)

Let T be NIP. Suppose for some singular $\kappa > |T|$, $2^\kappa = \kappa^+$. TFAE:

1. *T is not distal.*
2. *T has exact saturation at κ.*
3. *T has exact saturation in any μ such that $2^\mu = \mu^+$ and $\mu > |T|$.*

Idea:

If T is distal do something similar to DLO.

Otherwise, start with a non-distal indiscernible sequence I, and define
D-types similarly to the stable case. The point is that by distality a limit
type of a cut in I is not a D-type hence not realized.

Problem: we are now working on finding a "simple" example of a theory
which has singular compactness. "simple" here means, e.g., NSOP.

NTP$_2$ theories

Definition

A theory has the *tree property of the second kind* (TP$_2$) if there is a formula $\varphi(x, y)$, $k < \omega$ and $\langle a_{i,j} \mid i, j < \omega \rangle$ such that:

- Every row is k-inconsistent: for every $i < \omega$, $\{\varphi(x, a_{i,j}) \mid j < \omega\}$ is k-inconsistent (every k element set is inconsistent).
- Every vertical path is consistent: for every $\eta : \omega \to \omega$, $\left\{\varphi\left(x, a_{i,\eta(i)}\right) \mid i < \omega\right\}$ is consistent.
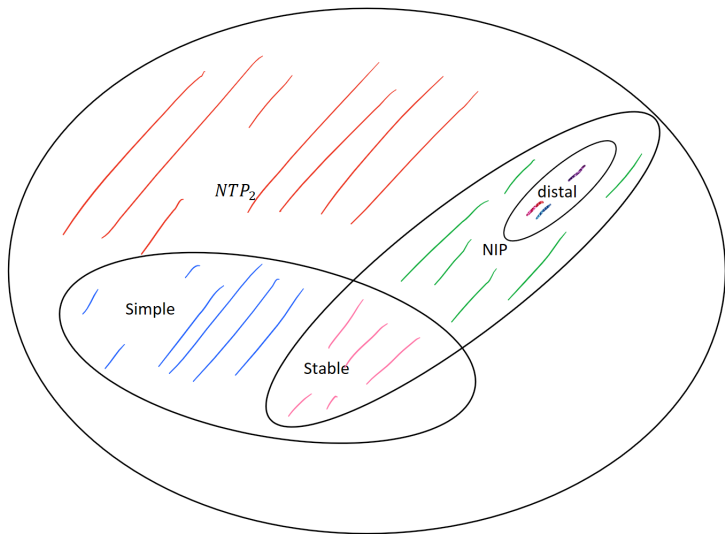
Definitions

T is NTP$_2$ if it does not have TP$_2$.

Examples

Every simple and NIP theory, Ultraproduct of the *p*-adics, ordered random graph.

A map of the known universe

Forking

Conjecture: Suppose that T is NTP_2. Then under mild set theoretical assumptions, such as square $+$ CH for κ (e.g., $V = L$ and $\operatorname{cof}(\kappa) > |T|$), T has exact saturation at κ iff T is not distal.

Forking Independence is one of the most important notions in model theory. Understanding what it is, or, sometimes trying to find suitable replacements is a recurring theme.

The idea is to find a notion of independence such as linear independence, that will allow us to analyze models.

In simple theories, forking behaves extremely nicely, and this was the main tool in proving the theorem for simple theories.

Forking

Definition

- A formula $\varphi(x, a)$ *divides* over A if there is a sequence $\langle a_i \mid i < \omega \rangle$ starting with a which is indiscernible over A and such that $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent.

- A formula $\varphi(x, a)$ *forks* over A if it belongs to the ideal of formulas generated by dividing formulas, in other words, if there are $\psi_i(x, b_i)$ for $i < n$ such that $\varphi(x, a) \vdash \bigvee_{i<n} \psi_i(x, b_i)$ and $\psi(x, b_i)$ divides over A.

Notation: we write $a \underset{A}{\bigcup} B$ for: the type $\mathrm{tp}(a/BA)$ does not fork over A (no formula from there forks).

Forking in NTP_2

Some properties of forking in NTP_2 have been known for some time.

Fact (w/ Artem Chernikov)

Forking equals dividing over models in NTP_2: if $\varphi(x, b)$ forks over a model M, then it divides over it.

Fact (Itai Ben Yaacov and Artem Chernikov)

An independence theorem for NTP_2 theories holds:
Suppose that T is NTP_2 and that M is a model. Assume that $c \mathop{\smile}\limits_{M} ab$, $a \mathop{\smile}\limits_{M} bb'$ and $b \equiv_M b'$. Then there is some c' such that $c'a \equiv_M ca$, $c'b' \equiv_M cb$ and $c' \mathop{\smile}\limits_{M} ab'$.

This theorem implies the usual independence theorem for simple theories which states that if $a \mathop{\smile}\limits_{M} b$ then given $c_1 \equiv_M c_2$ with $c_1 \mathop{\smile}\limits_{M} a$, $c_2 \mathop{\smile}\limits_{M} b$ then there is some c such that $ca \equiv_M c_1a$, $cb \equiv_M c_2b$ and $c \mathop{\smile}\limits_{M} ab$. This was crucial in the proof of the simple case.

Resilient theories

With Pierre Simon we have a new theorem about forking in a subclass of NTP_2 called *resilient*.

Definition

T is not *reslient* if there is some formula $\varphi(x, y)$, and some indiscernible sequence $\langle a_i \mid i \in \mathbb{Z} \rangle$ such that $\varphi(x, a_0)$ divides over $a_{\neq 0}$ but $\{\varphi(x, a_i) \mid i \in \mathbb{Z}\}$ is consistent.

Fact

Every NIP and simple theory is resilient, and every resilient theory is NTP_2.

It is an open question whether every NTP_2 theory is resilient.

A new theorem about forking in a subclass of NTP_2

Recall the definition of the independence property.

Definition

A theory is NIP (No Independence Property; also, dependent) if for no formula $\varphi(x, y)$ are there $\langle a_i \mid i < \omega \rangle$ and $\langle b_s \mid s \subseteq \omega \rangle$ such that $\varphi(a_i, b_s)$ iff $i \in s$.

Theorem (w/ Pierre Simon)

Suppose that T is resilient. Then the following is impossible:
There exists an infinite set A, a formula $\varphi(x, y)$ and some $k < \omega$ such that for every subset $s \subseteq A$, there is some b_s such that $\varphi(x, b_s)$ k-divides over $A \backslash s$ and for all $a \in s$, $a \models \varphi(x, b_s)$.

This theorem contributes to the feeling that NTP_2 is "NIP up-to non-forking".

Many more classes
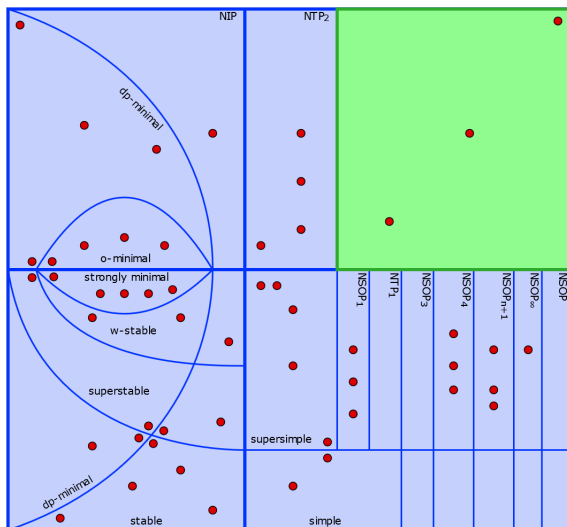
Due to Gabe Conant

What I didn't talk about

There are many more things one can say and I didn't touch at all. Here is a short list of interesting subjects.

1. NIP fields and groups maybe with extra properites (too many people to mention...).
   A lot is still unknown: what are stable fields? what can be said about NIP fields or subclasses such as strongly dependent fields? By work of Will Johnson we now know what are dp-minimal fields — the simplest kind of NIP fields, but the general results are scarce.

2. Measures and topological dynamics (Hrushovski, Pillay, Chernikov, Simon, Krupinski, ...).

3. Finite combinatorics and regularity lemmas (Chernikov, Strachenko, Malliaris, Shelah, ...).

4. Decomposition theorems of types. (Shelah, Simon, ...).

The end

Thank you!