

Spatial Logic of Tangled Closure and Derivative Operators

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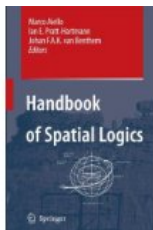
Joint work with Ian Hodkinson



Papers:

- Spatial logic of modal μ -calculus and tangled closure operators. *arXiv*
- The tangled derivative logic of the real line and zero-dimensional spaces. *Advances in Modal Logic*, vol. 11. www.aiml.net

What is *Spatial Logic* ?



*By a spatial logic, we understand any formal language interpreted over a class of structures featuring **geometrical** entities and relations, broadly construed.*

Basic modal language \mathcal{L}_\square

- a set of propositional **variables/atoms** p, q, \dots

- Boolean connectives:

$$\neg\varphi \quad \varphi \wedge \psi \quad \varphi \vee \psi \quad \varphi \rightarrow \psi \quad \varphi \leftrightarrow \psi$$

- box modality $\square\varphi$

- diamond modality $\diamond\varphi$ is $\neg\square\neg\varphi$

Kripke Semantics for \mathcal{L}_\square

Kripke frame: a directed graph $\mathcal{F} = (W, R)$ with $R \subseteq W \times W$.

Successor set: $R(x) = \{y : xRy\}$

A **model** on \mathcal{F} : assigns to each formula φ a **truth set** $\llbracket \varphi \rrbracket \subseteq W$.

Truth at a point: $x \models \varphi$ means $x \in \llbracket \varphi \rrbracket$.

Semantic conditions:

$$\llbracket \neg\varphi \rrbracket = W \setminus \llbracket \varphi \rrbracket$$

$$\llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket,$$

$$x \models \Box\varphi \text{ iff } R(x) \subseteq \llbracket \varphi \rrbracket$$

$$x \models \Diamond\varphi \text{ iff } R(x) \cap \llbracket \varphi \rrbracket \neq \emptyset.$$

$$\therefore \llbracket \Diamond\varphi \rrbracket = R^{-1}\llbracket \varphi \rrbracket \quad !!!!$$

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Truth of φ in a model: means that $\llbracket \varphi \rrbracket = W$.

This is **first-order** definable in the structure

$$(W, R, \{\llbracket p \rrbracket : p \text{ is an atom}\})$$

by the sentence $\forall x \varphi^*(x)$, where

$$(\Box \varphi)^*(x) \text{ is } \forall y (xRy \rightarrow \varphi^*(y))$$

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- Means that φ is true in every model on \mathcal{F} .

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Logic of a space $X := \{\varphi : \varphi \text{ is valid in } X\}$

- The logic of any space includes S4.
- The logic of any **separable dense-in-itself metric** space is exactly S4. This includes the Euclidean spaces \mathbb{R}^n for all $n \geq 1$, the rationals \mathbb{Q} , Cantor space, Baire space, . . .
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C.I. Lewis 1932

$\varphi \rightarrow \psi$ defined as $\neg \Diamond(\varphi \wedge \neg \psi)$

S4 defined as S1+ $\Diamond \Diamond \varphi \rightarrow \Diamond \varphi$

S1 AXIOMS

$$(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$$

$$(\varphi \wedge \psi) \rightarrow \varphi$$

$$\varphi \rightarrow (\varphi \wedge \varphi)$$

$$((\varphi \wedge \psi) \wedge \chi) \rightarrow (\varphi \wedge (\psi \wedge \chi))$$

$$\varphi \rightarrow \neg \neg \varphi$$

$$((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \chi)) \rightarrow (\varphi \rightarrow \chi)$$

$$(\varphi \wedge (\varphi \rightarrow \psi)) \rightarrow \psi$$

RULES

uniform substitution for atoms

$$\frac{\varphi, \psi}{\varphi \wedge \psi}$$

$$\frac{\varphi, \varphi \rightarrow \psi}{\psi}$$

$$\frac{(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi), \chi}{\chi(\psi/\varphi)}$$

Standard definition of S4

To a suitable basis for non-modal propositional calculus add the axioms

$$\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$$

$$\Box\varphi \rightarrow \varphi$$

$$\Box\varphi \rightarrow \Box\Box\varphi$$

and rule
$$\frac{\varphi}{\Box\varphi}$$

This is due to Gödel 1933



with $\Box\varphi$ written as $B\varphi$ “ φ is provable”.

Equivalent to Lewis' system with Becker's additional axiom

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Frames for S4:

$\mathcal{F} = (W, R)$ validates S4 iff R is reflexive and transitive (a **quasi-order**).

$\Box\varphi \rightarrow \varphi$ corresponds to reflexivity.

$\Box\varphi \rightarrow \Box\Box\varphi$ corresponds to transitivity.

In any S4-frame, the collection

$$\{R(x) : x \in W\}$$

is a basis for the **Alexandroff** topology on W , in which

$$\text{cl}(S) = R^{-1}(S)$$

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Tarski 1938 : Sentential calculus and topology

- Gave a topological interpretation of connectives that validates intuitionistic logic:

$\llbracket p \rrbracket = \text{any open set}$

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- Showed that the logic of any *dissectable* space is exactly the intuitionistic calculus.
- Included a proof that any separable dense-in-itself metric space is dissectable.

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Tarski's Original Dissection Theorem:

Let X be a dense-in-itself normal topological space with a countable basis of open sets.

Let G be a non-empty open subset of X , and let $r < \omega$.

Then G can be partitioned into non-empty subsets

$$G_1, \dots, G_r, B$$

such that the G_i 's are all open and

$$\text{cl}(G) \setminus G \subseteq \text{cl} B \subseteq \text{cl} G_1 \cap \dots \cap \text{cl} G_r.$$

[Proof credited to Samuel Eilenberg]



McKinsey and Tarski



1944 *The Algebra of Topology*

- Defined a **closure algebra** as a Boolean algebra with a unary operation Cx having

$$x \leq Cx = CCx$$

$$C(x + y) = Cx + Cy$$

$$C0 = 0$$

- Showed: if X a *dissectable* space, any **finite** closure algebra embeddable into the closure algebra of subsets of some open subset of X .

Works for any *dissectable* closure algebra in place of (the closure algebra of) X .



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The modal mu-calculus language $\mathcal{L}_{\square}^{\mu}$

Allows formation of the least fixed point formula

$$\mu p.\varphi$$

when p is positive in φ .

The greatest fixed point formula $\nu p.\varphi$ is

$$\neg \mu p.\varphi(\neg p/p).$$

Semantics in a model on a frame or space:

$\llbracket \mu p.\varphi \rrbracket$ is the **least** fixed point of the operator $S \mapsto \llbracket \varphi \rrbracket_{p:=S}$

$$\llbracket \mu p.\varphi \rrbracket = \bigcap \{S \subseteq W : \llbracket \varphi \rrbracket_{p:=S} \subseteq S\}$$

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The **tangle modality** language $\mathcal{L}_{\square}^{\langle t \rangle}$

Allows formation of the formula

$$\langle t \rangle \Gamma$$

when Γ is any finite non-empty set of formulas.

Semantics of $\langle t \rangle$ in a model on a frame:

$x \models \langle t \rangle \Gamma$ iff there is an **endless R -path**

$$x R x_1 \cdots x_n R x_{n+1} \cdots \cdots$$

in W with each member of Γ being true at x_n for infinitely many n .

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Cluster analysis

A transitive frame (W, R) is a partially ordered set of **clusters**, equivalence classes under the relation

$$x \equiv y \quad \text{iff} \quad x = y \text{ or } xRyRx.$$

Put $C_x = \{y : x \equiv y\}$, and lift R to a partial order of clusters by

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If the frame is **finite**, an endless R -path must eventually enter some **non-degenerate** cluster and stay there.

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$\langle t \rangle \Gamma$ is definable in $\mathcal{L}_{\square}^{\mu}$

In any model on a **transitive** frame,

$$\llbracket \langle t \rangle \Gamma \rrbracket = \bigcup \{ S \subseteq W : S \subseteq \bigcap_{\gamma \in \Gamma} R^{-1}(\llbracket \gamma \rrbracket \cap S) \}$$

i.e. $\llbracket \langle t \rangle \Gamma \rrbracket$ is the largest set S such that

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But $R^{-1}\llbracket \varphi \rrbracket = \llbracket \diamond \varphi \rrbracket$, and \bigcap interprets \bigwedge ,
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Origin of the tangle modality:

van Benthem 1976

The bisimulation-invariant fragment of first-order logic is equivalent to \mathcal{L}_{\square} .

This holds relative to any elementary class of frames (e.g. transitive)
And relative to the class of finite frames [Rosen 1997]

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Origin of the tangle modality:

van Benthem 1976

The bisimulation-invariant fragment of first-order logic is equivalent to \mathcal{L}_{\square} .

This holds relative to any elementary class of frames (e.g. transitive)
And relative to the class of finite frames [Rosen 1997]

Janin & Walukiewicz 1993

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Fernández-Duque 2011

- coined the name “tangle”.
- axiomatised the $\mathcal{L}_{\square}^{\langle t \rangle}$ -logic of the class of all (finite) S4-frames, as S4 +

$$\text{Fix: } \langle t \rangle \Gamma \rightarrow \diamond(\gamma \wedge \langle t \rangle \Gamma), \quad \text{all } \gamma \in \Gamma.$$

$$\text{Ind: } \square(\varphi \rightarrow \bigwedge_{\gamma \in \Gamma} \diamond(\gamma \wedge \varphi)) \rightarrow (\varphi \rightarrow \langle t \rangle \Gamma).$$

- provided its topological interpretation, with closure in place of R^{-1} .

The **derivative** modality language $\mathcal{L}_{[d]}$

Replace \Box and \Diamond by $[d]$ and $\langle d \rangle$, with $\llbracket \langle d \rangle \varphi \rrbracket = R^{-1} \llbracket \varphi \rrbracket$

Define $\Box \varphi$ as $\varphi \wedge [d]\varphi$, and $\Diamond \varphi = \varphi \vee \langle d \rangle \varphi$.

In a topological space X , the **derivative** of a subset S is

$$\text{deriv } S = \{x \in X : x \text{ is a limit point of } S\}.$$

$x \in \text{deriv } S$ iff every neighbourhood O of x has $(O \setminus \{x\}) \cap S \neq \emptyset$.

In a model on X , $\llbracket \langle d \rangle \varphi \rrbracket = \text{deriv} \llbracket \varphi \rrbracket$, so

$x \models \langle d \rangle \varphi$ iff every punctured neighbourhood of x intersects $\llbracket \varphi \rrbracket$,

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$\mathcal{L}_{[d]}$ is more expressive than \mathcal{L}_{\square}

- $\llbracket \square \varphi \rrbracket$ = the **interior** of $\llbracket \varphi \rrbracket$. $\llbracket \diamond \varphi \rrbracket$ = the **closure** of $\llbracket \varphi \rrbracket$.
- Validity of the R -transitivity axiom

$$4 : \quad \langle d \rangle \langle d \rangle \varphi \rightarrow \langle d \rangle \varphi$$

holds iff X is a **T_D space**, meaning $\text{deriv}\{x\}$ is always closed.
[Aull & Thron 1962]

- Validity of the axiom

$$D : \quad \langle d \rangle \top$$

holds iff X is **dense-in-itself**, i.e. no isolated points.

Validity of D in a **frame** holds iff R is total: $\forall x \exists y (xRy)$.

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Shehtman 1990:

Derived sets in Euclidean spaces and modal logic.

Proved

- the $\mathcal{L}_{[d]}$ -logic of every **zero-dimensional** separable dense-in-itself metric space is KD4.
- the $\mathcal{L}_{[d]}$ -logic of the Euclidean space \mathbb{R}^n for any $n \geq 2$ is

$$\text{KD4} + \mathbf{G}_1 : \langle d \rangle p \wedge \langle d \rangle \neg p \rightarrow \langle d \rangle (\diamond p \wedge \diamond \neg p)$$

Conjectured

- the $\mathcal{L}_{[d]}$ -logic of the real line \mathbb{R} is $\text{KD4} + \mathbf{G}_2$, where \mathbf{G}_n is

$$\bigwedge_{i \leq n} \langle d \rangle Q_i \rightarrow \langle d \rangle \left(\bigwedge_{i \leq n} \diamond \neg Q_i \right), \quad \text{with } Q_i = p_i \wedge \bigwedge_{i \neq j \leq n} \neg p_j.$$

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Replace $\langle t \rangle$ by $\langle dt \rangle$.

Interpret $\langle dt \rangle$ by replacing R^{-1} by deriv:

In a model on space X ,

$$\begin{aligned} \llbracket \langle dt \rangle \Gamma \rrbracket &= \text{the tangled derivative of } \{\llbracket \gamma \rrbracket : \gamma \in \Gamma\}. \\ &= \bigcup \{S \subseteq X : S \subseteq \bigcap_{\gamma \in \Gamma} \text{deriv}(\llbracket \gamma \rrbracket \cap S)\}. \end{aligned}$$

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Defining $\langle t \rangle$ from $\langle dt \rangle$

In a topological space X , $\langle t \rangle \Gamma$ is equivalent to

$$(\wedge \Gamma) \vee \langle d \rangle (\wedge \Gamma) \vee \langle dt \rangle \Gamma$$

if, and only if X is a T_D space.

Axioms for logics:

Let L be any logic in some language.

- L_t is the extension of L by the tangle axioms

$$\text{Fix: } \langle dt \rangle \Gamma \rightarrow \langle d \rangle (\gamma \wedge \langle dt \rangle \Gamma)$$

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- $L.U$ is the extension of L that has the universal modality \forall with semantics

$$x \models \forall \varphi \text{ iff for all } y \in W, y \models \varphi,$$

the S5 axioms and rules for \forall , and the axiom $\forall \varphi \rightarrow [d]\varphi$.

- $L.UC$ is the extension of $L.U$ for which C is the axiom

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Let X be any dense-in-itself metric space

language	logic complete over X	sound over X
$\mathcal{L}_{\square}^{(t)}$	S4 <i>t</i>	yes
$\mathcal{L}_{\square\forall}$	S4.UC	if X connected
$\mathcal{L}_{\square\forall}^{(t)}$	S4 <i>t</i> .UC	if X connected
$\mathcal{L}_{[d]}$	KD4G ₁ ¹	if G ₁ valid in X
$\mathcal{L}_{[d]}^{(dt)}$	KD4G ₁ <i>t</i>	if G ₁ valid in X
$\mathcal{L}_{[d]\forall}$	KD4G ₁ .UC	if X connected & validates G ₁
$\mathcal{L}_{[d]\forall}^{(dt)}$	KD4G ₁ <i>t</i> .UC	if X connected & validates G ₁

¹answers Shehtman's question

The Case of \mathbb{R} :

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The Zero-Dimensional case:

Let X be any zero-dimensional dense-in-itself metric space.

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Strong completeness: ‘consistent sets are satisfiable’

- Let L be $KD4G_1$ or $KD4G_1t$ or $S4t$. Then any **countable** L -consistent set of formulas is satisfiable in any dense-in-itself metric space.
- Any countable $KD4t$ -consistent set of formulas is satisfiable in any **zero-dimensional** dense-in-itself metric space.

Can fail for frame and spatial semantics for “large enough” sets:

$$\{\Diamond p_i : i < \kappa\} \cup \{\neg\Diamond(p_i \wedge p_j) : i < j < \kappa\}$$

Not satisfiable in frame \mathcal{F} if $\kappa > \text{card } \mathcal{F}$.

Not satisfiable in space X if $\kappa > 2^{\text{card } X}$.

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Also shows that in the **canonical** model for $K4t$, the 'Truth Lemma' fails.

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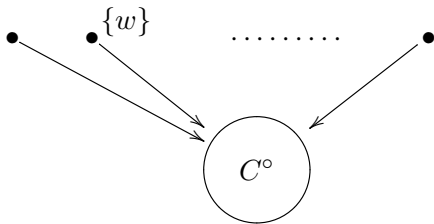
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Proving a logic L is complete over space X :

- 1 Prove the finite model property for L over Kripke frames:
if $L \not\vdash \varphi$, then φ is falsifiable in some suitable finite frame $\mathcal{F} \models L$.
- 2 Construct a surjective **d-morphism** $f : X \twoheadrightarrow \mathcal{F}$:

$$f^{-1}(R^{-1}(S)) = \text{deriv } f^{-1}(S).$$

Such an f preserves validity of formulas from X to \mathcal{F} , so $X \not\models \varphi$.



Modified Tarski Dissection Theorem

Let X be a dense-in-itself metric space.

Then X is **dissectable** as follows:

Let \mathbb{G} be a non-empty open subset of X , and let $r, s < \omega$.

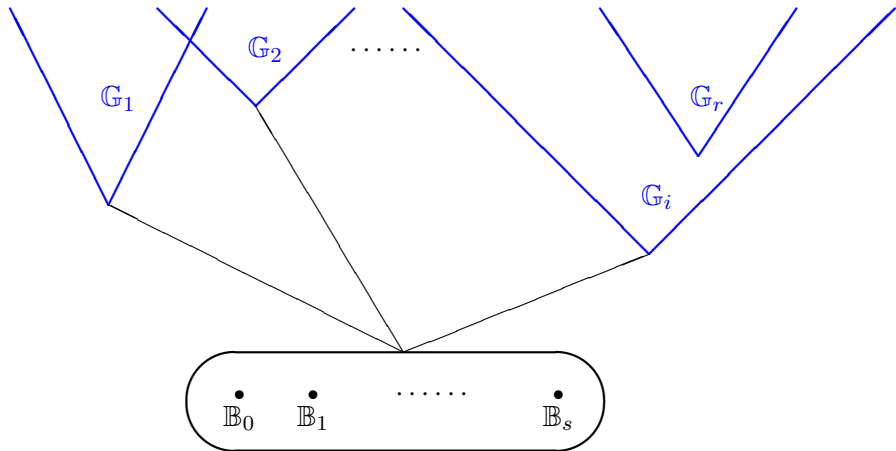
Then \mathbb{G} can be partitioned into non-empty subsets

$$\mathbb{G}_1, \dots, \mathbb{G}_r, \mathbb{B}_0, \dots, \mathbb{B}_s$$

such that the \mathbb{G}_i 's are all open and

$$\text{cl}(\mathbb{G}_i) \setminus \mathbb{G}_i = \text{deriv}(\mathbb{B}_j) = \text{cl}(\mathbb{G}) \setminus (\mathbb{G}_1 \cup \dots \cup \mathbb{G}_r).$$

This encodes a d-morphism $\mathbb{G} \rightarrow \mathcal{F}$,
if \mathcal{F} is a point-generated S4-frame.



Further dissections of a dense-in-itself metric X

- 1 Let \mathbb{G} be a non-empty open subset of X , and let $k < \omega$. Then there are pairwise disjoint non-empty subsets $\mathbb{I}_0, \dots, \mathbb{I}_k \subseteq \mathbb{G}$ satisfying

$$\text{deriv } \mathbb{I}_i = \text{cl}(\mathbb{G}) \setminus \mathbb{G} \quad \text{for each } i \leq k.$$

- 2 Let X be zero-dimensional.

If \mathbb{G} is a non-empty open subset of X , and $n < \omega$, then \mathbb{G} can be partitioned into non-empty open subsets $\mathbb{G}_0, \dots, \mathbb{G}_n$ such that

$$\text{cl}(\mathbb{G}_i) \setminus \mathbb{G}_i = \text{cl}(\mathbb{G}) \setminus \mathbb{G} \quad \text{for each } i \leq n.$$