Spatial Logic of Tangled Closure and Derivative Operators

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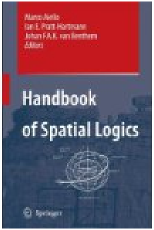
Joint work with Ian Hodkinson

Papers:

- Spatial logic of modal mu-calculus and tangled closure operators. *arXiv*
- The tangled derivative logic of the real line and zero-dimensional spaces. *Advances in Modal Logic, vol. 11.* `www.aiml.net`

What is *Spatial Logic* ?



By a spatial logic, we understand any formal language interpreted over a class of structures featuring geometrical entities and relations, broadly construed.

Basic modal language $\mathcal{L}_\Box$

- a set of propositional variables/atoms $p, q, \ldots$
- Boolean connectives:

$$\neg\varphi \qquad \varphi \wedge \psi \qquad \varphi \vee \psi \qquad \varphi \to \psi \qquad \varphi \leftrightarrow \psi$$

- box modality $\quad \Box\varphi$
- diamond modality $\quad \Diamond\varphi$ is $\neg\Box\neg\varphi$

Kripke Semantics for $\mathcal{L}_\Box$

Kripke frame: a directed graph $\mathcal{F} = (W, R)$ with $R \subseteq W \times W$.

Successor set: $R(x) = \{y : xRy\}$

A model on $\mathcal{F}$: assigns to each formula φ a truth set $[\![\varphi]\!] \subseteq W$.

Truth at a point: $x \models \varphi$ means $x \in [\![\varphi]\!]$.

Semantic conditions:

$$[\![\neg\varphi]\!] = W \setminus [\![\varphi]\!]$$

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!],$$

$$x \models \Box\varphi \quad \text{iff} \quad R(x) \subseteq [\![\varphi]\!]$$

$$x \models \Diamond\varphi \quad \text{iff} \quad R(x) \cap [\![\varphi]\!] \neq \emptyset.$$

$$\therefore \quad [\![\Diamond\varphi]\!] = R^{-1}[\![\varphi]\!] \qquad !!!!$$

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Truth of φ in a model: means that $[\![\varphi]\!] = W$.

This is first-order definable in the structure

$$(W, R, \{[\![p]\!] : p \text{ is an atom}\})$$

by the sentence $\forall x \varphi^*(x)$, where

$$(\Box\varphi)^*(x) \text{ is } \forall y(xRy \to \varphi^*(y))$$
$$(\Diamond\varphi)^*(x) \text{ is } \exists y(xRy \land \varphi^*(y))$$

Validity of φ in frame $\mathcal{F}$:

- Means that φ is true in every model on $\mathcal{F}$.

  i.e. φ is true at every point in every model on $\mathcal{F}$.

- This is monadic-second-order definable in $\mathcal{F}$ by $\forall p_1 \cdots \forall p_n \forall x \varphi^*(x)$

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Topological semantics for $\mathcal{L}_\Box$

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[\![\Box\varphi]\!] &= \mathrm{int}[\![\varphi]\!], \quad \text{the interior of } [\![\varphi]\!] \\
\therefore \quad [\![\Diamond\varphi]\!] &= \mathrm{cl}[\![\varphi]\!], \quad \text{the closure of } [\![\varphi]\!]
\end{aligned}$$

$x \models \Box\varphi$ iff there is an open set O with $x \in O \subseteq [\![\varphi]\!]$.

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Logic of a space $X := \{\varphi : \varphi$ is valid in $X\}$

- The logic of any space includes S4.
- The logic of any separable dense-in-itself metric space is exactly S4. This includes the Euclidean spaces $\mathbb{R}^n$ for all $n \geq 1$, the rationals $\mathbb{Q}$, Cantor space, Baire space,…
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C.I. Lewis 1932

$\varphi \mathrel{-\!\!3} \psi$ defined as $\neg\Diamond(\varphi \wedge \neg\psi)$

S4 defined as S1+ $\Diamond\Diamond\varphi \mathrel{-\!\!3} \Diamond\varphi$

S1 AXIOMS	RULES
$(\varphi \wedge \psi) \mathrel{-\!\!3} (\psi \wedge \varphi)$	uniform substitution for atoms
$(\varphi \wedge \psi) \mathrel{-\!\!3} \varphi$	
$\varphi \mathrel{-\!\!3} (\varphi \wedge \varphi)$	$\dfrac{\varphi,\ \ \psi}{\varphi \wedge \psi}$
$((\varphi \wedge \psi) \wedge \chi) \mathrel{-\!\!3} (\varphi \wedge (\psi \wedge \chi))$	
$\varphi \mathrel{-\!\!3} \neg\neg\varphi$	$\dfrac{\varphi,\ \ \varphi \mathrel{-\!\!3} \psi}{\psi}$
$((\varphi \mathrel{-\!\!3} \psi) \wedge (\psi \mathrel{-\!\!3} \chi)) \mathrel{-\!\!3} (\varphi \mathrel{-\!\!3} \chi)$	
$(\varphi \wedge (\varphi \mathrel{-\!\!3} \psi)) \mathrel{-\!\!3} \psi$	$\dfrac{(\varphi \mathrel{-\!\!3} \psi) \wedge (\psi \mathrel{-\!\!3} \varphi),\ \chi}{\chi(\psi/\varphi)}$

Standard definition of S4

To a suitable basis for non-modal propositional calculus add the axioms

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$
$$\Box\varphi \to \varphi$$
$$\Box\varphi \to \Box\Box\varphi$$

and rule $\dfrac{\varphi}{\Box\varphi}$

This is due to Gödel 1933

with $\Box\varphi$ written as $B\varphi$ "φ is provable".

Equivalent to Lewis' system with Becker's additional axiom

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Frames for S4:

$\mathcal{F} = (W, R)$ validates S4 iff R is reflexive and transitive (a quasi-order).

$\Box\varphi \to \varphi$ corresponds to reflexivity.
$\Box\varphi \to \Box\Box\varphi$ corresponds to transitivity.

In any S4-frame, the collection

$$\{R(x) : x \in W\}$$

is a basis for the Alexandroff topology on W, in which

$$\mathrm{cl}(S) = R^{-1}(S)$$

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Tarski 1938 : Sentential calculus and topology

- Gave a topological interpretation of connectives that validates intuitionistic logic:

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[\![p]\!] &= \text{any open set} \\
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- Showed that the logic of any *dissectable* space is exactly the intuitionistic calculus.
- Included a proof that any separable dense-in-itself metric space is dissectable.

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Tarski's Original Dissection Theorem:

Let X be a dense-in-itself normal topological space with a countable basis of open sets.

Let $\mathbb{G}$ be a non-empty open subset of X, and let $r < \omega$.

Then $\mathbb{G}$ can be partitioned into non-empty subsets

$$\mathbb{G}_1, \ldots, \mathbb{G}_r, \mathbb{B}$$

such that the $\mathbb{G}_i$'s are all open and

$$\mathrm{cl}(\mathbb{G}) \setminus \mathbb{G} \subseteq \mathrm{cl}\,\mathbb{B} \subseteq \mathrm{cl}\,\mathbb{G}_1 \cap \ldots \cap \mathrm{cl}\,\mathbb{G}_r.$$

[Proof credited to Samuel Eilenberg]

McKinsey and Tarski

1944 *The Algebra of Topology*

- Defined a closure algebra as a Boolean algebra with a unary operation $\mathsf{C}x$ having
$$x \le \mathsf{C}x = \mathsf{CC}x$$
$$\mathsf{C}(x+y) = \mathsf{C}x + \mathsf{C}y$$
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- Showed: if X a *dissectable* space, any finite closure algebra embeddable into the closure algebra of subsets of some open subset of X.

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The modal mu-calculus language $\mathcal{L}^{\mu}_{\Box}$

Allows formation of the least fixed point formula

$$\mu p.\varphi$$

when p is positive in φ.

The greatest fixed point formula $\nu p.\varphi$ is

$$\neg \mu p.\varphi(\neg p/p).$$

Semantics in a model on a frame or space:

$[\![\mu p.\varphi]\!]$ is the least fixed point of the operator $S \mapsto [\![\varphi]\!]_{p:=S}$

$$[\![\mu p.\varphi]\!] = \bigcap\{S \subseteq W : [\![\varphi]\!]_{p:=S} \subseteq S\}$$

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The tangle modality language $\mathcal{L}_{\Box}^{\langle t\rangle}$

Allows formation of the formula

$$\langle t\rangle\Gamma$$

when Γ is any finite non-empty set of formulas.

Semantics of $\langle t\rangle$ in a model on a frame:

$x \models \langle t\rangle\Gamma$ iff there is an endless R-path

$$xRx_1 \cdots x_n R x_{n+1} \cdots\cdots$$

in W with each member of Γ being true at x_n for infinitely many n.

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Cluster analysis

A transitive frame (W, R) is a partially ordered set of clusters, equivalence classes under the relation

$$x \equiv y \quad \text{iff} \quad x = y \text{ or } xRyRx.$$

Put $C_x = \{y : x \equiv y\}$, and lift R to a partial order of clusters by

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If the frame is finite, an endless R-path must eventually enter some non-degenerate cluster and stay there.

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so $\langle t\rangle\Gamma$ has the same meaning as the $\mathcal{L}^{\mu}_{\Box}$-formula

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Suggests a topological semantics: replace R^{-1} by closure

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Origin of the tangle modality:

van Benthem 1976

The bisimulation-invariant fragment of first-order logic is equivalent to $\mathcal{L}_\Box$.

This holds relative to any elementary class of frames (e.g. transitive)
And relative to the class of finite frames [Rosen 1997]

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over the class of finite transitive frames, the bisimulation-invariant fragment of monadic second-order logic collapses to that of first-order logic, with both fragments, and $\mathcal{L}^\mu_\Box$, being equivalent to the tangle extension $\mathcal{L}^{\langle t \rangle}_\Box$.

Origin of the tangle modality:

van Benthem 1976

The bisimulation-invariant fragment of first-order logic is equivalent to $\mathcal{L}_\Box$.

This holds relative to any elementary class of frames (e.g. transitive)
And relative to the class of finite frames [Rosen 1997]

Janin & Walukiewicz 1993

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over the class of finite transitive frames, the bisimulation-invariant fragment of monadic second-order logic collapses to that of first-order logic, with both fragments, and $\mathcal{L}_\Box^\mu$, being equivalent to the tangle extension $\mathcal{L}_\Box^{\langle t \rangle}$.

Fernández-Duque 2011

- coined the name "tangle".
- axiomatised the $\mathcal{L}_{\Box}^{\langle t \rangle}$-logic of the class of all (finite) S4-frames, as S4 +

  Fix: $\langle t \rangle \Gamma \to \Diamond(\gamma \wedge \langle t \rangle \Gamma)$, all $\gamma \in \Gamma$.

  Ind: $\Box(\varphi \to \bigwedge_{\gamma \in \Gamma} \Diamond(\gamma \wedge \varphi)) \to (\varphi \to \langle t \rangle \Gamma)$.

- provided its topological interpretation, with closure in place of R^{-1}.

The derivative modality language $\mathcal{L}_{[d]}$

Replace $\Box$ and $\Diamond$ by $[d]$ and $\langle d \rangle$, with $[\![\langle d \rangle\varphi]\!] = R^{-1}[\![\varphi]\!]$

Define $\boxdot\varphi$ as $\varphi \wedge [d]\varphi$, and $\diamondsuit\varphi = \varphi \vee \langle d \rangle\varphi$.

In a topological space X, the derivative of a subset S is

$$\operatorname{deriv} S = \{x \in X : x \text{ is a limit point of } S\}.$$

$x \in \operatorname{deriv} S$ iff every neighbourhood O of x has $(O \setminus \{x\}) \cap S \neq \emptyset$.

In a model on X, $[\![\langle d \rangle\varphi]\!] = \operatorname{deriv}[\![\varphi]\!]$, so

$x \models \langle d \rangle\varphi$ iff every punctured neighbourhood of x intersects $[\![\varphi]\!]$,

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$\mathcal{L}_{[d]}$ is more expressive than $\mathcal{L}_{\Box}$

- $[\![\Box\varphi]\!]$ = the interior of $[\![\varphi]\!]$. $[\![\Diamond\varphi]\!]$ = the closure of $[\![\varphi]\!]$.
- Validity of the R-transitivity axiom

$$4: \quad \langle d\rangle\langle d\rangle\varphi \to \langle d\rangle\varphi$$

holds iff X is a T_D space, meaning $\mathrm{deriv}\{x\}$ is always closed. [*Aull & Thron* 1962]

- Validity of the axiom

$$\mathrm{D}: \quad \langle d\rangle\top$$

holds iff X is dense-in-itself, i.e. no isolated points.

Validity of D in a frame holds iff R is total: $\forall x\exists y(xRy)$.

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Shehtman 1990:
Derived sets in Euclidean spaces and modal logic.

Proved

- the $\mathcal{L}_{[d]}$-logic of every zero-dimensional separable dense-in-itself metric space is KD4.
- the $\mathcal{L}_{[d]}$-logic of the Euclidean space $\mathbb{R}^n$ for any $n \geq 2$ is

$$\text{KD4} + \text{G}_1 : \langle d \rangle p \wedge \langle d \rangle \neg p \to \langle d \rangle (\Diamond p \wedge \Diamond \neg p)$$

Conjectured

- the $\mathcal{L}_{[d]}$-logic of the real line $\mathbb{R}$ is KD4 + G_2, where G_n is

$$\bigwedge_{i \leq n} \langle d \rangle Q_i \to \langle d \rangle \big(\bigwedge_{i \leq n} \Diamond \neg Q_i \big), \qquad \text{with } Q_i = p_i \wedge \bigwedge_{i \neq j \leq n} \neg p_j.$$

[Proven later by Shehtman, and by Lucero-Bryan]

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The tangled derivative language $\mathcal{L}_\Box^{\langle dt\rangle}$

Replace $\langle t\rangle$ by $\langle dt\rangle$.
Interpret $\langle dt\rangle$ by replacing R^{-1} by deriv:

In a model on space X,

$$\begin{aligned}[\![\langle dt\rangle\Gamma]\!] &= \text{the tangled derivative of } \{[\![\gamma]\!] : \gamma \in \Gamma\}. \\ &= \bigcup\{S \subseteq X : S \subseteq \bigcap_{\gamma\in\Gamma} \mathrm{deriv}([\![\gamma]\!] \cap S)\}.\end{aligned}$$

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Defining $\langle t \rangle$ from $\langle dt \rangle$

In a topological space X, $\langle t \rangle \Gamma$ is equivalent to

$$(\bigwedge \Gamma) \vee \langle d \rangle (\bigwedge \Gamma) \vee \langle dt \rangle \Gamma$$

if, and only if X is a T_D space.

Axioms for logics:

Let L be any logic in some language.

- Lt is the extension of L by the tangle axioms

  Fix: $\langle dt\rangle\Gamma \to \langle d\rangle(\gamma \wedge \langle dt\rangle\Gamma)$

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- L.U is the extension of L that has the universal modality $\forall$ with semantics

  $$x \models \forall\varphi \text{ iff for all } y \in W,\ y \models \varphi,$$

  the S5 axioms and rules for $\forall$, and the axiom $\forall\varphi \to [d]\varphi$.

- L.UC is the extension of L.U for which C is the axiom

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Let X be any dense-in-itself metric space

language	logic complete over X	sound over X
$\mathcal{L}_{\Box}^{\langle t\rangle}$	$\mathrm{S4}t$	yes
$\mathcal{L}_{\Box\forall}$	$\mathrm{S4.UC}$	if X connected
$\mathcal{L}_{\Box\forall}^{\langle t\rangle}$	$\mathrm{S4}t\mathrm{.UC}$	if X connected
$\mathcal{L}_{[d]}$	$\mathrm{KD4G_1}$ [1]	if $\mathrm{G_1}$ valid in X
$\mathcal{L}_{[d]}^{\langle dt\rangle}$	$\mathrm{KD4G_1}t$	if $\mathrm{G_1}$ valid in X
$\mathcal{L}_{[d]\forall}$	$\mathrm{KD4G_1.UC}$	if X connected & validates $\mathrm{G_1}$
$\mathcal{L}_{[d]\forall}^{\langle dt\rangle}$	$\mathrm{KD4G_1}t\mathrm{.UC}$	if X connected & validates $\mathrm{G_1}$

[1] answers Shehtman's question

The Case of $\mathbb{R}$:

- The $\mathcal{L}_{[d]}^{\langle dt \rangle}$-logic of $\mathbb{R}$ is KD4G$_2t$.
- The $\mathcal{L}_{[d]\forall}^{\langle dt \rangle}$-logic of $\mathbb{R}$ is KD4G$_2t$.UC.

The Zero-Dimensional case:

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Strong completeness: 'consistent sets are satisfiable'

- Let L be $KD4G_1$ or $KD4G_1t$ or $S4t$. Then any countable L-consistent set of formulas is satisfiable in any dense-in-itself metric space.
- Any countable $KD4t$-consistent set of formulas is satisfiable in any zero-dimensional dense-in-itself metric space.

Can fail for frame and spatial semantics for "large enough" sets:

$$\{\Diamond p_i : i < \kappa\} \cup \{\neg\Diamond(p_i \wedge p_j) : i < j < \kappa\}$$

Not satisfiable in frame $\mathcal{F}$ if $\kappa > \operatorname{card}\mathcal{F}$.

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$$\Sigma = \{\Diamond p_0\} \cup \\ \{\Box(p_{2n} \to \Diamond(p_{2n+1} \wedge q)), \Box(p_{2n+1} \to \Diamond(p_{2n+2} \wedge \neg q)) : n < \omega\}$$

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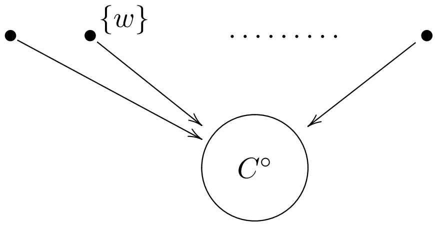
Proving a logic L is complete over space X:

1. Prove the finite model property for L over Kripke frames: if L $\nvdash \varphi$, then φ is falsifiable in some suitable finite frame $\mathcal{F} \models$L.
2. Construct a surjective d-morphism $f : X \twoheadrightarrow \mathcal{F}$:

$$f^{-1}(R^{-1}(S)) = \operatorname{deriv} f^{-1}(S).$$

Such an f preserves validity of formulas from X to $\mathcal{F}$, so $X \not\models \varphi$.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C°

Modified Tarski Dissection Theorem

Let X be a dense-in-itself metric space.
Then X is dissectable as follows:

Let $\mathbb{G}$ be a non-empty open subset of X, and let $r, s < \omega$.

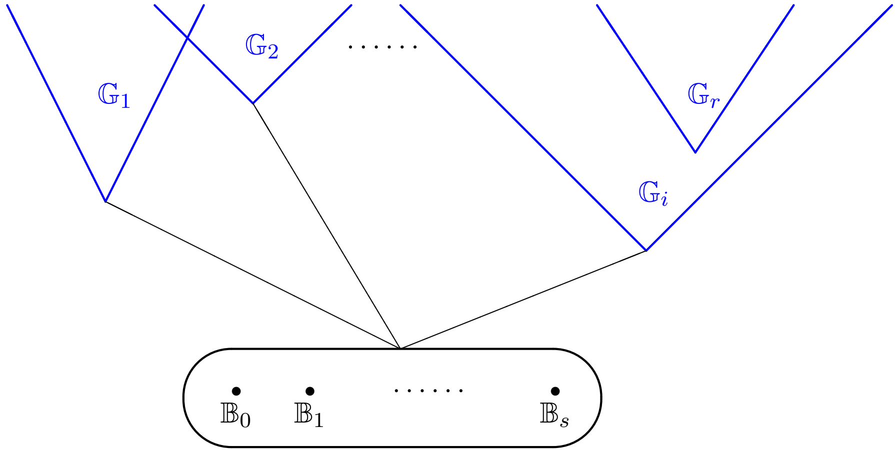
Then $\mathbb{G}$ can be partitioned into non-empty subsets

$$\mathbb{G}_1, \ldots, \mathbb{G}_r, \mathbb{B}_0, \ldots, \mathbb{B}_s$$

such that the $\mathbb{G}_i$'s are all open and

$$\mathrm{cl}(\mathbb{G}_i) \setminus \mathbb{G}_i = \mathrm{deriv}(\mathbb{B}_j) = \mathrm{cl}(\mathbb{G}) \setminus (\mathbb{G}_1 \cup \cdots \cup \mathbb{G}_r).$$

This encodes a d-morphism $\mathbb{G} \twoheadrightarrow \mathcal{F}$,
if $\mathcal{F}$ is a point-generated S4-frame.



Further dissections of a dense-in-itself metric X

1. Let $\mathbb{G}$ be a non-empty open subset of X, and let $k < \omega$. Then there are pairwise disjoint non-empty subsets $\mathbb{I}_0, \ldots, \mathbb{I}_k \subseteq \mathbb{G}$ satisfying

$$\operatorname{deriv} \mathbb{I}_i = \operatorname{cl}(\mathbb{G}) \setminus \mathbb{G} \quad \text{for each } i \leq k.$$

2. Let X be zero-dimensional.
   If $\mathbb{G}$ is a non-empty open subset of X, and $n < \omega$, then $\mathbb{G}$ can be partitioned into non-empty open subsets $\mathbb{G}_0, \ldots, \mathbb{G}_n$ such that

$$\operatorname{cl}(\mathbb{G}_i) \setminus \mathbb{G}_i = \operatorname{cl}(\mathbb{G}) \setminus \mathbb{G} \text{ for each } i \leq n.$$