Randomized algorithms in computability theory

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Logic Colloquium, Leeds, UK

August 1st, 2016

1. How useful is randomness?

How useful is randomness? (1)

Whether having access to a ‘random source’ can help use achieve more than what we could do without is perhaps one of the most fundamental questions in theoretical computer science.

P: Class of languages which can be decided in (deterministic) polynomial time.

BPP: Class of languages which can be decided in polynomial time if given access to a random source, with probability, say, 0.99.

Open question: Does P = BPP?

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Theorem (De Leeuw-Moore-Shannon-Shapiro, 1956)

Let X be an infinite binary sequence (or language). If there is an algorithm (machine) Φ with access to a random source R (also a sequence of bits) such that

$$Pr[\Phi(R) = X] > 0,$$

then X is computable.

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Proof. If the probability is $> 1/2$, find the value of each bit of X by a 'majority vote'. If not, apply the Lebesgue density theorem to get a relative probability $> 1/2$ and do the same.

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$\mathcal{A}$ is easier (not harder) to solve than $\mathcal{B}$ if we can computably get *some* solution of $\mathcal{A}$ from *any* solution of $\mathcal{B}$.

Non-uniform version, noted $\mathcal{A} \leq_w \mathcal{B}$:

$$\text{for every } X \in \mathcal{B} \text{ there is } \Phi \text{ such that } \Phi(X) \in \mathcal{A}$$

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$\mathbf{DNC}_{\mathrm{bis}}$: functions $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{K}(f(n)) > n$.

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• Much more interesting is the case of $\mathbf{HI}$ (functions $f : \mathbb{N} \to \mathbb{N}$ not dominated by any computable one)... We will come back to it later.

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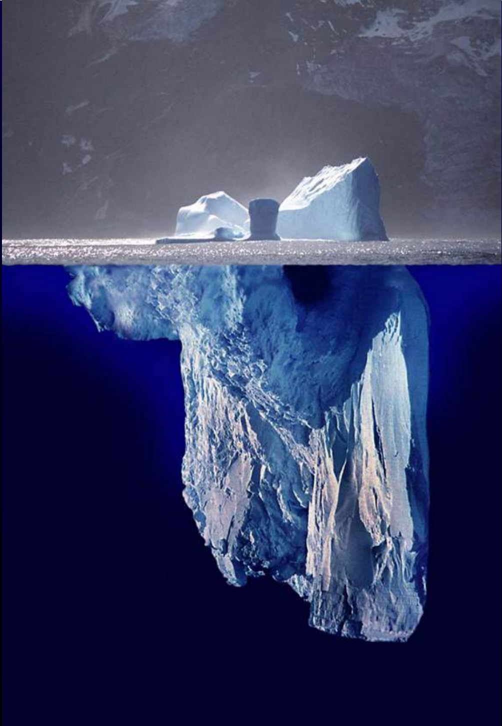
Then we can apply a version of 'majority vote': for each n, wait until $\Psi(R)(n)$ returns a value for $2/3$ of R's. Take $g(n)$ = the maximum value seen over all those R's. Then g is computable and for every n:

$$Pr[\Psi(R)(n) > g(n)] \leq 1/3$$

and by Fatou's lemma,

$$Pr[\Psi(R) \text{ dominates } g] \leq 1/3$$

2. Randomness vs depth



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(A later improvement by F. Stephan: if R is Martin-Löf random and computes a member of $\mathbf{PA}$, then R computes the halting problem $\emptyset'$).

Measuring the difficulty

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There is a **universal probabilistic algorithm** Ξ, that is, for any probabilistic algorithm Φ, for some constant c all every class $\mathcal{C}$ of finite and infinite objects:

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Levin's coding theorem: $\mathbf{M}(\{x\}) = 2^{-\mathrm{K}(x)} \cdot O(1)$

Measuring the difficulty (2)

Let $\mathbf{PA}_n$ be the set of coherent finite theories of arithmetical formulas such that for every formula ψ encodable in n bits, ψ or $\neg\psi$ is in the theory (think of $\mathbf{PA}_n$ as the set of strings of length 2^n which can be extended into a member of $\mathbf{PA}$).

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For any $x \in \mathbf{PA}_n, \quad \mathbf{I}(x : \emptyset') \gtrsim n - O(1)$

This is better since, for Φ a randomized algorithm and Z a finite or infinite object,

$$\mathbb{E}\left(2^{\mathbf{I}(\Phi(R):Z)}\right) = O(1)$$

(Zvonkin-Levin's information conservation theorem)

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In particular, for x physically obtainable, $\mathbf{I}(x : \emptyset')$ is small. On the other hand completions of PA have high common information with $\emptyset'$ (Levin's theorem). Thus they cannot be physically obtainable!

Deep classes (1)

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Definition (B., Porter)
Let $\mathcal{P}$ be a Π^0_1 class. Let $\mathcal{P}_n$ be a set of finite strings of length n which can be extended to an element of $\mathcal{P}$. We say that $\mathcal{P}$ is **deep** if

$$\mathbf{M}(\mathcal{P}_n) \leq \frac{1}{h(n)}$$

for some computable h which tends to infinity.

Deep classes (2)

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Theorem (B., Porter)

The following Π^0_1 classes are deep:

- Levin complex sequences: binary sequences such that $\mathrm{K}(X_n \ldots X_{n+k}) \geq 0.9\,k$ for all n and all $k \geq c$ (**after Rumyantsev, Khan**).
- $\mathbf{DNC}_q$, with $\prod_n (1 - 1/q(n)) = 0$: functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(e) \neq \phi_e(e)$ and $f(e) \leq q(e)$ (**after Miller**).
- Sequences of sets $(F_0, F_1, \ldots)$ where F_i is a finite set of strings of length i, and $card(F_i) \geq f(i)$ for some computable non-decreasing f tending to ∞.
- ...

Deep classes (3)

The interesting thing is that even when the corresponding mass problem is easier than $\mathbf{PA}$, a deep Π^0_1 class 'behaves like $\mathbf{PA}$' in its interactions with randomness. For example:

Theorem (B., Porter)

- If R is Martin-Löf random and does not compute $\emptyset'$, then R does not compute any element of a deep Π^0_1 class (Stephan for $\mathbf{PA}$).
- This remains true for $R \oplus A$, when A is K-trivial (Miller-Day for $\mathbf{PA}$).

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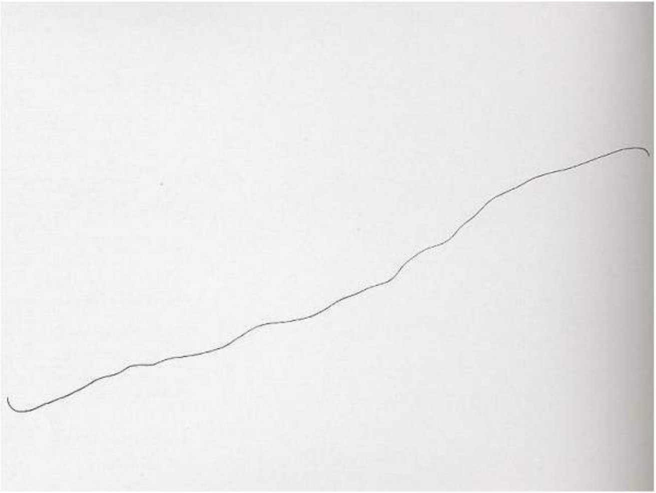
This relates to an old idea due to Bennett, who argued that Kolmogorov complexity captures the idea of ‘information’, but not of ‘depth’.

Why ‘deep’? (2)



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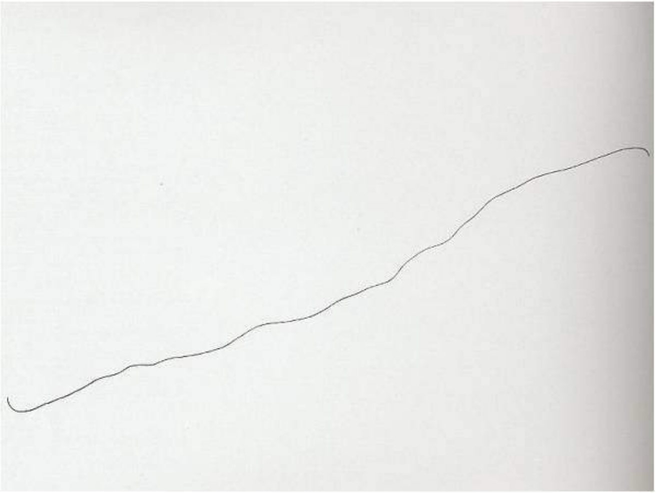

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non-random / compressible... but deep!

Logical depth

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Definition (after Bennett 1988)

An infinite binary sequence X is *logically deep* if for every computable time bound (function) T,

$$\mathrm{K}^T(X_0 \ldots X_n) - \mathrm{K}(X_0 \ldots X_n) \to \infty$$

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(It is not true however that a Π^0_1 class whose members are all logically deep must be deep).

3. When randomness helps

Getting a hyperimmune function ‘at random’ (1)

We come back to the mass problem

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This one **does** admit a probabilistic algorithm, due to Kautz (1991), and clarified by Gács and Shen (2012).

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However, there is a probabilistic way to do this, via a **fireworks argument**.

Fireworks (1)

Suppose we walk into a fireworks shop.

- The fireworks sold there are very cheap so we are suspicious that some of them are defective.
- Since they are cheap we can ask the owner to test a few of them before buying one.
- **Our goal: either buy a good one (untested) and take it home OR get the owner to fail a test, and then sue him.**

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This works because the only bad case is when $k+1$ is the position of the first bad box.

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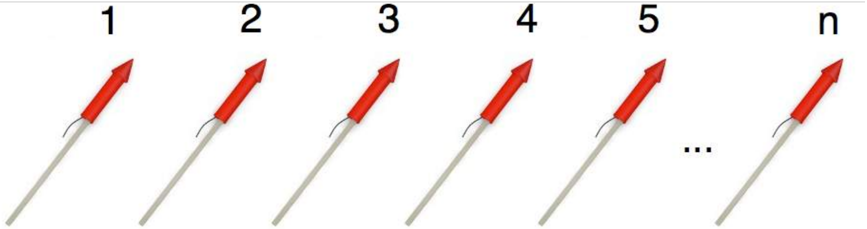

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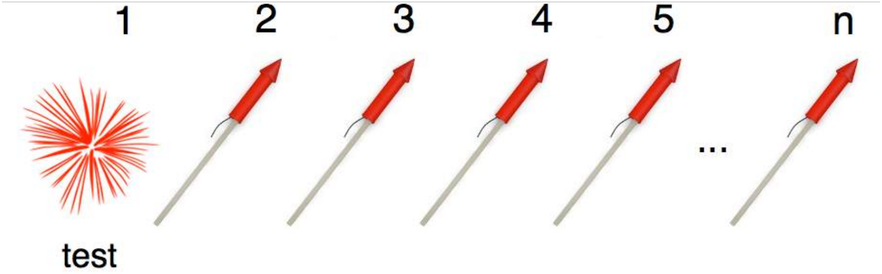

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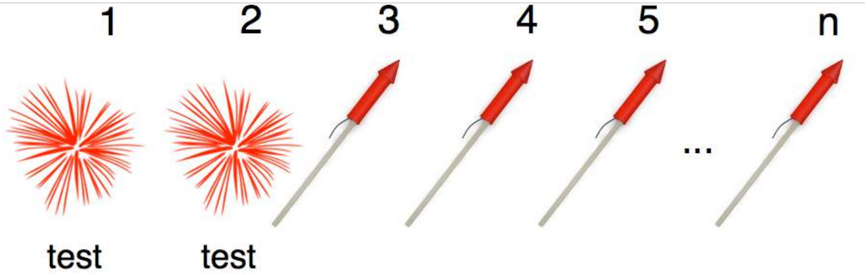

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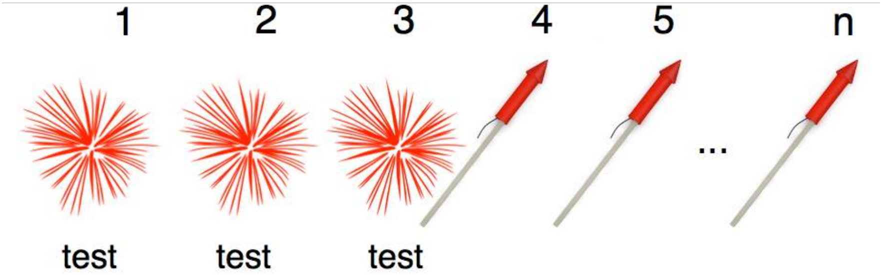

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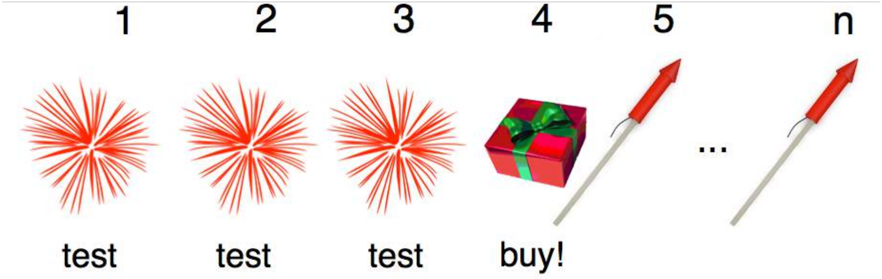

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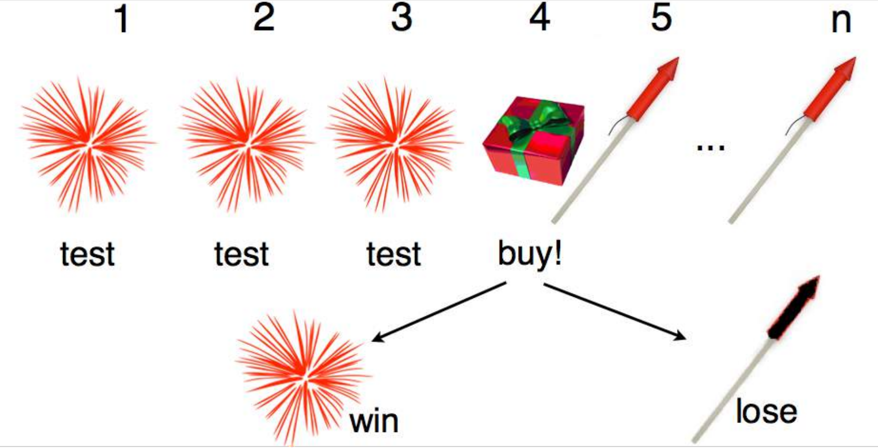

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In fact, Kautz showed: **Every sequence which is Martin-Löf random relative to $\emptyset'$ computes a function in $\mathbf{HI}$.**

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Question: are there other types of non-trivial probabilistic algorithms which could apply to computability theory? (currently no other known type)

Turning De Leeuw et al's theorem around

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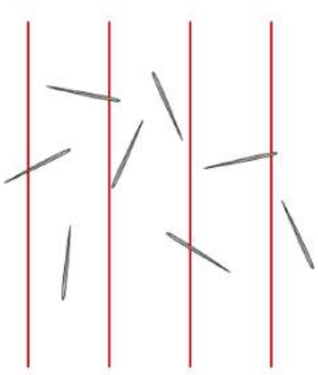
Buffon's needle shows that π is a computable number ;-)
(one can use the needle to get a probabilistic algorithm to compute π, thus by De Leeuw et al's theorem π is computable)



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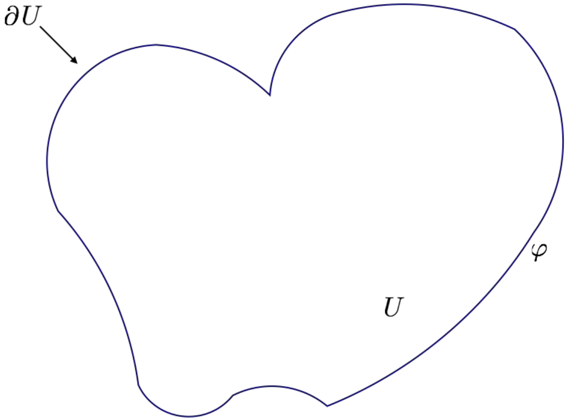

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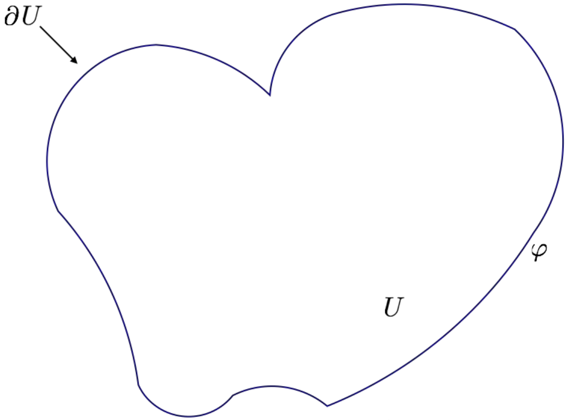

$$\text{Dirichlet's problem: find } f \text{ s.t.} \begin{cases} f \text{ continuous on } \bar{U} \\ \Delta f = 0 \\ f = \varphi \text{ on } \partial U \end{cases}$$

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Is this solution computable? (= when ∂U, φ are computable, is f computable?)

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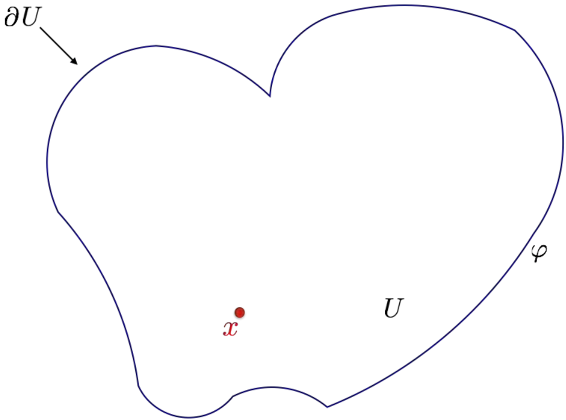

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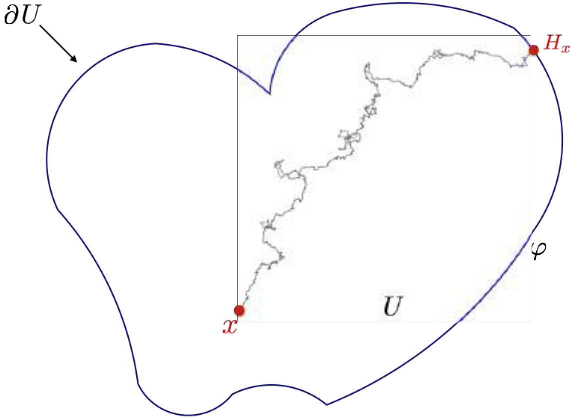

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Theorem (Allen-B.-Slaman)
Given x and a random source R, one can compute a Brownian path starting from x and compute its first intersection with ∂U.

Thus, we have a probabilistic algorithm to compute $f(x)$ given x!

Thus f is computable! (by De Leeuw et al's theorem, essentially)

In conclusion...

S. Barry Cooper (1943-2015)

Thank you, Barry !