Computable Model Theory: Part 3

Uri Andrews

University of Wisconsin

August 2016
Leeds, UK

Recall:

Because this is the only setting that I know where computable/recursive and decidable do not mean the same thing, I want to start with clarifying:

Definition

A structure M is computable if it has domain (a computable subset of) ω and its atomic diagram is a computable set (in particular, we assume the language is computable as well).
Example: $(\mathbb{N}, +, \cdot)$

Definition

A structure M is decidable if it has domain (a computable subset of) ω and its elementary diagram is a computable set.
Example: $(\mathbb{Q}, <)$.

Scott Sets

We will use countable Scott sets constantly in this talk.

Definition

A Scott set $\mathcal{S}$ is a nonempty subset of $P(\omega)$ with the following properties:

- If $X \in \mathcal{S}$ and $Y \leq_T X$ then $Y \in \mathcal{S}$
- If $X, Y, \in \mathcal{S}$ then $X \oplus Y \in \mathcal{S}$.
- If $T \subseteq 2^{<\omega}$ is infinite and $T \leq_T X \in \mathcal{S}$, then there is some $Y \in \mathcal{S}$ which is a path through T.

Definition

$E \subseteq \omega$ is an enumeration of the Scott set $\mathcal{S}$ if $\{E_i \mid i \in \omega\} = \mathcal{S}$ where $E_i = \{j \mid \langle i, j \rangle \in E\}$.

Observation

For any X in a Scott set $\mathcal{S}$, there is a $Y >_T X$ in $\mathcal{S}$.

My favorite theorem

Definition

E is an effective enumeration of the Scott set $\mathcal{S}$ if E is an enumeration of $\mathcal{S}$ and there are computable functions witnessing the closure properties of the Scott set. i.e. There is a computable function $f(i,j)$ so that if T is an infinite tree in $2^{<\omega}$ and $T = \varphi_i(E_j)$, then $E_{f(i,j)}$ is a path through T.

One of my main goals of this talk is to highlight this theorem:

Theorem (Marker, '83)

If X computes an enumeration of the Scott set $\mathcal{S}$, then X also computes an effective enumeration of the Scott set $\mathcal{S}$.

Effectivity for free! This is a statement that no computability theorist believes at first sight. After all, how could I possibly know which of the infinitely many columns is a path through T?

Coded reals in models of PA

Definition

If $a \in M \models \mathrm{PA}$, then $r(a) = \{n \mid \text{the } n^{\text{th}} \text{ prime divides } a\}$.
If $M \models \mathrm{PA}$, we let $\mathcal{SS}(M) = \{r(a) \mid a \in M\}$.

Theorem (Scott-Tennenbaum)

If M is a nonstandard model of PA, then $\mathcal{SS}(M)$ is a Scott set.

Proof.

Let T be an infinite tree coded by $r(a)$. That is, $\sigma \in T$ if and only if $p_\sigma | a$ (the σth prime). Then consider the set of n so that $\{\sigma \mid p_\sigma | a\}$ defines a tree up to length n and there is a string σ_n of length n coded by some number b so that $p_{\sigma_n} | a$. This is a definable set and includes every standard integer. By overspill/induction, there is a nonstandard integer in this set. □

M computes an enumeration of $\mathcal{SS}(M)$

Theorem (Tennenbaum)

There is no computable nonstandard model of PA.

Proof.

Let M be a model of PA with universe ω. Then we define $E = \{\langle i, a\rangle \in \omega \mid p_i | a\}$. Then E is computable in (the atomic diagram of) M, but E is an enumeration of a Scott set, so it is not computable. We call this E the Standard Enumeration of $\mathcal{SS}(M)$. □

Theorem (Solovay)

If M is non-standard, then the Standard Enumeration of $\mathcal{SS}(M)$ is M-effective. i.e., M can compute the functions witnessing the closedness of the Scott set. So, M computes an effective enumeration of $\mathcal{SS}(M)$.

The proof uses an awesome trick.

Δ_1^0-definable sets are computable in M

Lemma

Let M be a model of PA. Let A be a Δ_1^0-definable set. i.e A and its complement are Σ_1^0-definable. Then A is computable in M.

Proof.

The novel problem in this lemma is that $\forall x < v$ is not obviously computable to check if v is a non-standard integer. In $\mathbb{N}$, bounded quantification is obviously computable: Just check the finitely many cases needed. But here, there are infinitely many $x < v$.

Awesome Trick

Matiyasevich's theorem shows that in any model of PA, any Σ_1^0 set X is diophantine. That is, there is a polynomial $p(\bar{x}, \bar{y})$ so that $\bar{x} \in X$ if and only if there is a $\bar{y}$ so that $p(\bar{x}, \bar{y}) = 0$.

But the definition of a diophantine set has an existential quantifier, but no bounded exponentiation. So it is r.e.

Proof of Solovay's theorem

Proof.

Given $a \in M$ (where we think of a as coding the the tree T), there is some $\sigma \in 2^{<M}$ and $v \in M$ so that

$$(\forall n < |\sigma|)\, p_{\sigma|n} | a \wedge$$

$$(\forall \tau \in 2^{|\sigma|+1}) \neg (\forall n < |\tau|)\, p_{\tau|n} | a \wedge$$

$$(\forall i < |\sigma|)(p_i | v \leftrightarrow \sigma(i) = 1)$$

Using the Lemma about Δ^0_1 definable sets, we see that it is M-computable to verify that (σ, v) is a witness. So, we M-computably just output such a witness. □

Note the overspill argument still appears here: There must be a longest length of a p_σ so that all of its initial segments divides a (by induction). This longest length cannot be finite.

Models of PA are computably boundedly saturated

Theorem (Friedman)

Let $\bar{a} \in M \models \mathrm{PA}$. Let $p(x, \bar{a})$ be a computable set of Σ_n formulas which is consistent with M. Then p is realized in M.

Proof.

We use the Σ_n satisfaction predicate Sat_n. Let $\{\varphi_i(x, \bar{a}) \mid i \in \omega\}$ be a computable enumeration of $p(x, \bar{a})$. Using the fact that computable sets are representable in PA, there is a formula

$$\psi(a, w) := \mathrm{Sat}_n\Big(\bigwedge_{i \leq a} \varphi_i(w)\Big).$$

Then, since $p(c, \bar{a})$ is consistent, for every $n \in \omega$, $M \models \exists w \psi(n, w)$. By overspill, we get a w satisfying all of $p(w, \bar{a})$. □

Some coded sets

Corollary

If $M \models \mathrm{PA}$ is non-standard, then for each $n \in \omega$, $\mathrm{Th}(M) \cap \Sigma_n \in \mathcal{SS}(M)$.

Proof.

Consider the partial type $p(x) := \{p_i | x \leftrightarrow \mathrm{Sat}_n(i) \mid i \in \omega\}$. By the saturation, p is realized. $\square$

Corollary (Feferman)

If $M \equiv \mathbb{N}$ is nonstandard, then M computes every arithmetical set.

$\mathcal{S}$-saturation

Definition

Let $\mathcal{S} \subseteq P(\omega)$. M is $\mathcal{S}$-saturated if

- Every type $p(\bar{x})$ realized in M is computable in some $X \in \mathcal{S}$.
- If $p(\bar{x}, \bar{y})$ is a type computable in some $X \in \mathcal{S}$ and $\bar{a} \in M$ is so that $p(\bar{x}, \bar{a})$ is consistent, then p is realized in M.

Usually model-theorists consider full saturation, i.e., $\mathcal{S} = P(\omega)$.

Lemma

If $\mathcal{S}$ is a countable Scott set and T is a complete theory in $\mathcal{S}$, then T has a countable $\mathcal{S}$-saturated model. If M and N are both countable $\mathcal{S}$-saturated models of T, then $M \cong N$.

How computable is an $\mathcal{S}$-saturated model?

Theorem (Marker, using Goncharov and Peretyat'kin)

If M is $\mathcal{S}$-saturated and E is an enumeration of $\mathcal{S}$, then there is a copy of M whose elementary diagram is computable in E (i.e. M is E-decidable).

This uses the following result of Goncharov and Peretyat'kin:

Theorem (Goncharov and Peretyat'kin, 78)

Let A be ω-homogeneous. Let E be a d-enumeration of the types realized in A. Suppose further that E has the d-effective extension property. Then there is a copy of A which is d-decidable.

Definition

An enumeration E of types has the d-effective extension property if there is a d-computable function $g(i,j)$ so that if $p(\bar{x}) = E_i$ and $\varphi(\bar{x}, y) = E_j$ is consistent with p, then $E_{g(i,j)}$ is a type containing p and φ.

The implication

Given a type $p(\bar{x})$ and a formula $\varphi(\bar{x}, y)$ it is computable in the enumeration of the T-types in $\mathcal{S}$ to produce a type containing both. The only difficulty is in finding an index for it. But we can do this easily if we produce a new enumeration of the T-types in $\mathcal{S}$ wherein we explicitly build many sets just to be these "extension types". Then, since $\mathcal{S}$ is closed under Turing reduction (since it is a Scott set), we have that this is also an enumeration of the T-types in $\mathcal{S}$.

So, starting with a d-computable enumeration of $\mathcal{S}$, we produce a d-computable enumeration of the T-types in $\mathcal{S}$ and then a second d-computable enumeration of the T-types in $\mathcal{S}$ with the d-effective extension property, then apply Goncharov/Peretyat'kin.

My favorite theorem - reminder slide

One of my main goals of this talk is to highlight this theorem.

Definition

E is an effective enumeration of the Scott set $\mathcal{S}$ if E is an enumeration of $\mathcal{S}$ and there are computable functions witnessing the closure properties of the Scott set. i.e. There is a computable function $f(i,j)$ so that if T is an infinite tree in $2^{<\omega}$ and $T = \varphi_i(E_j)$, then $E_{f(i,j)}$ is a path through T.

Theorem (Marker, '83)

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Effectivity for free! This is a statement that no computability theorist believes at first sight. After all, how could I possibly know which of the infinitely many columns is a path through T?

Marker's proof

Proof.

Let E be the X-computable enumeration of $\mathcal{S}$. There is some completion T of PA contained in $\mathcal{S}$. Then X also computes (by the last theorem) the elementary diagram of an $\mathcal{S}$-saturated model M of T.

Now, let $R = \mathcal{SS}(M)$. This R is an X-effective enumeration of a Scott set. We now only need:

Lemma

If M is an $\mathcal{S}$-saturated model of PA, then $\mathcal{S} = \mathcal{SS}(M)$.

Proof.

$\mathcal{SS} \subseteq \mathcal{S}$, since $r(a)$ is computable in $\mathrm{tp}(a)$. $\mathcal{S} \subseteq \mathcal{SS}$, since for any set $A \in \mathcal{S}$, we can cook up a type in $\mathcal{S}$ containing all the formulae $p_i | x$ if and only if $i \in A$. $\square$

$\square$

Some general comments

Marker's theorem does not mention PA or model theory, yet the only known proof involves specifically looking at models of PA and looking at $\mathcal{S}$-saturated models.

It would be fascinating to see if there were a purely computability-theoretic proof. If so, what serves the role of homogeneity/saturation?

It follows from results of Lachlan and Soare that there is an enumeration X of a jump ideal $\mathcal{S}$ so that X does not compute any enumeration of $\mathcal{S}$ which can find jumps effectively. So, this property of "free" uniformity is unique for finding paths through trees.

Post-Talk added Slide: Uniformity in Marker's Result

Many thanks to Arno Pauly for pointing this out.

Theorem

This can be made uniform. That is, there is a single Turing reduction Φ so that given any enumeration X of a Scott set $\mathcal{S}$, $\Phi(X)$ gives an effective enumeration of the Scott set $\mathcal{S}$

Proof.

Start by guessing that the first column is a completion of PA and begin the algorithm described in the above proof. If you find that it is not, then just make all the columns you started be codes for finite sets (put all 0's in those columns from here on), and move on to guessing that the second column is a completion of PA. Eventually, you settle on a column that works and you produce the effective enumeration of the Scott set. Note that the first finitely many columns code finite sets, so in particular, they do not code infinite trees, so we don't need to find paths through them.

Solovay's Theorem (context for Marker's result)

Both of these results are unpublished, though Knight wrote a wonderful expository paper "True approximations and models of arithmetic" explaining the proofs.

Theorem (Solovay, '82 (unpublished))

A set X can compute a non-standard model of $\mathrm{Th}(\mathbb{N}, +, \cdot)$ if and only if it can compute an effective enumeration of a Scott set $\mathcal{S}$ which contains every arithmetical set.

Note that we gave one direction of the proof above, and the other is another "worker" construction. For general $T \supset \mathrm{PA}$:

Theorem (Solovay, 91 (unpublished))

Fix $T \supset \mathrm{PA}$ complete. A set X can compute a non-standard model of T if and only if X computes an enumeration R of a Scott set $\mathcal{S}$ so that $T \cap \exists_n \in \mathcal{S}$ for each n and there are a sequence of functions t_n, uniformly $\Delta^0_n(X)$ so that for all n, $\lim_s t_n(s)$ is an R-index for $T \cap \exists_n$, and for each s, $R_{t_n(s)} \subseteq T \cap \exists_n$.

Solovay theories (named for a feature of the last theorem)

This is an interesting feature, from a computable model theory perspective, of a theory:

Definition

A complete theory T is Solovay if $T \cap \exists_n$ is uniformly Σ^0_n.

Observation

If T has a computable model, then T is a Solovay theory.
There are only countably many Solovay theories, since they are all computable from $0^{(\omega)}$.

So, in some sense, Solovay theories looks at first blush like a rough analogue of "theory with a computable model".

How far are Solovay theories from having computable models?

Inspired by Solovay's work on models of PA, and working from the basic premise that nothing is worse than arithmetic:

Theorem (A.-Knight, '13)

There is a complete Solovay theory T extending PA so that the degrees which compute models of T are precisely the degrees which compute non-standard models of $\mathrm{Th}(\mathbb{N}, +, \cdot)$, i.e. those which compute Scott sets containing every arithmetical set.

Conversely:

Theorem (Solovay/Knight and A.-Cai-Diamondstone-Lempp-J.S.Miller (rediscovered))

If X computes an enumeration of a Scott set containing every arithmetical set and T is a Solovay theory, then X computes a model of T.

Again, the following slides...

were prepared for but were not covered during the Leeds tutorial series. I recognize that I had over-prepared. If anybody does in fact read the online version of these slides, I hope that they can still be of some use.

A general framework for these discussions

Definition

For a theory T, we say the degree spectrum of T is the set of Turing degrees which compute models of T.

We contrast this to the more commonly studied notion of the degree spectrum of a structure:

Definition

For a structure M, we say the degree spectrum of M is the set of Turing degrees which compute a copy of M.

Observation

For any theory T, $\mathrm{Spec}(T) = \bigcup_{M \models T} \mathrm{Spec}(M)$

An aside: These are in fact different

Theorem (A.-J.S.Miller, '15)

There are superstable theories so that $\mathrm{Spec}(T)$ is exactly the PA-degrees, $\mathrm{Spec}(T)$ is exactly the degrees which compute ML-random sets, and so that $\mathrm{Spec}(T)$ is exactly the union of 2 upper cones of c.e. sets.
None of these are degree spectra of structures.
Also, none of these are degree spectra of atomic theories.

Theorem (A.-J.S.Miller, '15)

The set of non-hyperarithmetical degrees is not the spectrum of a theory.

Theorem (Greenberg-Montalbàn-Slaman, '13)

The set of non-hyperarithmetical degrees is the spectrum of a structure.

Atomic (and even ω-stable) theories also have new spectra

Theorem (A.-Knight, '13)

There is a completion of PA whose spectrum is exactly the degrees which compute non-standard models of $\mathrm{Th}(\mathbb{N}, +, \cdot)$. This set is not the spectrum of any structure.[a]

[a] In most of these results, the difficult part is showing the negative direction.

Theorem (A.-Cai-Diamondstone-Lempp-J.S.Miller)

This spectrum is also not the spectrum of any ω-stable theory

Theorem (A.-Cai-Diamondstone-Lempp-J.S.Miller)

There is a spectrum of an ω-stable theory which is not the spectrum of any structure.

Some classes of theories reflected in degree spectra

Definition

Let $\mathcal{P} \subseteq \mathcal{Q}$ be two classes of theories. We say the difference between them is reflected in degree spectra if there is some $T \in \mathcal{Q}$ so that there is no $T' \in \mathcal{P}$ so that $\mathrm{Spec}(T') = \mathrm{Spec}(T)$.

Theorem (Summary of above)

The line between superstable and superstable atomic theories is reflected in degree spectra. The line between superstable atomic and ω-stable theories are reflected in degree spectra.

Question

What other dividing lines in model theory are reflected in degree spectra?

This is generally compelling because it asks something at the core of computable model theory: What computability-theoretic content is encoded in model theoretic properties?

The end

Thank you!