Computable Model Theory Part 2:

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Recall:

Because this is the only setting that I know where computable/recursive and decidable do not mean the same thing, I want to start with clarifying:

Definition

A structure M is computable if it has domain (a computable subset of) ω and its atomic diagram is a computable set (in particular, we assume the language is computable as well). Example: $(\mathbb{N}, +, \cdot)$

Definition

A structure M is decidable if it has domain (a computable subset of) ω and its elementary diagram is a computable set. Example: $(\mathbb{Q}, <)$.

Recall: Strongly minimal theories

Definition

A first order theory T is *strongly minimal* if every definable subset of every model is a finite or co-finite subset of the model.

Example

- A regular acyclic graph with finite valence (say, the theory of a Cayley graph of a finitely generated group);
- A vector space (say, the theory of $(\mathbb{Q}, +)$);
- An algebraically closed field, (say, the theory of $(\mathbb{C}, +, \cdot, 0, 1)$)

Recall: Algebraic closure and independence

Definition

For elements $\bar{a}, b \in M$, we say $b \in \operatorname{acl}(\bar{a})$ if there is a formula $\varphi(x, \bar{y})$ so that $\varphi(M, \bar{a})$ is finite and $M \models \varphi(b, \bar{a})$.
i.e., b is in a finite $\bar{a}$-definable set.

Definition

A set $S \subset M$ is independent if each $x \in S$ is not in $\operatorname{acl}(S \setminus \{x\})$.

Definition

For any n, there is a unique type of an independent n-tuple. We call this the generic n-type.

Recall: Dimension

Definition

- If M is a model, a maximal independent subset is called a basis for M.
- The dimension of a set X is the size of a maximal independent subset.

Fact

In strongly minimal theories, the above are well-defined. That is, if B_1 and B_2 are maximal independent subsets of X, then $|B_1| = |B_2|$.

Recall: Dimensions characterize models.

Recall: The Zilber trichotomy

Zilber conjectured that every strongly minimal theory was of one of three types:

- Disintegrated (Essentially binary)
- Locally Modular (Essentially a quasi-vector space)
- Field-like (Essentially an algebraically closed field)

Theorem (Hrushovski 1991)

The Zilber trichotomy is false. There are Hrushovski constructions which build non-trichotomous theories.

These structures are inherently combinatorial in nature, and have no algebraic content.

Recall: Questions

As a model theorist, the characterization of models by dimension seems extremely clear. What questions would a computability theorist ask here? And why?

Question

With such a clear understanding of how models work, what is the connection between computing the theory and computing a model? If I know the theory, can I put these ideas to work to produce its models? If I can compute the models, can I reconstruct the theory? And if one model is computable how non-computable can the other models be?

Question

Since all the models look the same, i.e., they are just closures of different sized independent sets, is it true that if one model is computable then other models are computable? If not, which sets of models can be the set of computable models?

Recall: More questions

Question

To each of the questions above, how does the geometry of the theory (i.e., whether it falls into the Zilber trichotomy and if so, under which category) affect the answer? Is everything as computable as can be for disintegrated theories? Modular theories?

Question

If we restrict the language to a finite signature, does that change anything?

Recall: Relative Computability of Models vs. Theories

Theorem (Goncharov-Harizanov-Laskowski-Lempp-McCoy, '03)

If T is a disintegrated strongly minimal theory and $M \models T$ is computable, then T and thus every other model of T is computable from $0''$.

Theorem (A.-Medvedev, '14)

If T is a strongly minimal locally modular theory expanding a group and $M \models T$ is computable, then T and thus every other model of T is computable from $0''$.

Theorem (A., 2013)

There is a strongly minimal theory T so that $T \equiv_T 0^{(\omega)}$ and every countable model of T is computable.

So, there's no hope for any similar argument to go through for general strongly minimal theories.

Relative computability of models

Theorem (A.-Knight)

Given T any strongly minimal theory with a computable model. If $M \models T$, then M has a copy computable in $0'''$.

The ideas for this theorem were informed to a great degree by this theorem of Lerman-Schmerl and Knight:

Theorem (Lerman-Schmerl '79, Knight '94)

If T is an (arithmetical) $\aleph_0$-categorical theory so that for all n, $T \cap \exists_{n+2}$ is Σ^0_{n+1}, then T has a computable model.

The idea is based on a so-called pull-down lemma, and the infinite worker versions there-of.

Lerman-Schmerl's Pull-down lemma

Lemma

If the $n+1$-quantifier diagram of $A \models T$ is d'-computable and $T \cap \exists_{n+2}$ is c.e. in d, then the n-quantifier diagram of some $B \cong A$ is d-computable.

Proof Sketch.

We do a finite-injury construction trying to build an isomorphism from our B to the given A. We use the c.e. set $T \cap \exists_{n+2}$ to ask whether what we are doing are reasonable. For every tuple $\bar{a} \in A$, A tells us (correctly) some $n+1$-quantifier formula which isolates its $n+1$-quantifier type. Why? Because T is $\aleph_0$-categorical, so there are only finitely many $|\bar{x}|$-types total, so in particular only finitely many $n+1$-quantifier types. $\square$

**Wave hands at how this proof is completed and how Knight "inducts in reverse" on ω

General structure of the proof, and one highlighted idea

Similarly, we have each "worker" $0^{(n)}$ building the n-quantifier diagram of a model. But for us, we have to be more specific, because no single formula will inform the worker below about the type of this element. So, rather, the nth worker decides on a pair $(\theta(\bar{a}), k)$ where $\theta(\bar{a})$ has Morley rank k and is of minimal Morley rank/degree formula in the n-quantifier type of $\bar{a}$. Once the lower worker knows it, that determines the type.

Definition

A structure M is boundedly saturated if whenever $p(x, \bar{y})$ is an n-quantifier-type for some n and $\bar{a} \in M$ is so $p(x, \bar{a})$ is consistent, then there is a realization of p in M.

Lemma (The Key Lemma)

Every model M of T is either boundedly saturated or not.

What does the Key Lemma buy us?

If M is boundedly saturated, this tells us that as we go about guessing at the n-quantifier diagram of a copy of M, whatever we build (consistent with T) is "safe", in that it exists somewhere in M. This is reassuring as we go about the infinite worker method necessary (with no top model).

The second case tells us that some n-quantifier type is omitted over a tuple. But this must be the unique non-algebraic n-quantifier type. Thus, for some n, every element is algebraic over a fixed finite set via an n-quantifier formula. We can use this to get ourselves a top model.

Both of these are very useful assumptions. I wonder whether this dichotomy will appear more in the future.

Some other results using pull-down lemmas

The following theorem uses 4 layers of a pull-down lemma:

Theorem (Downey-Jockusch '94, Thurber '95, Knight-Stob '00)

If A is a boolean algebra with a low_4 presentation, then A has a computable copy.

We do not know what happens at low_5 or higher.

Theorem (Marker-R.Miller)

Every low copy of $M \models \mathrm{DCF}_0$ has a computable copy.

Marker and Miller showed that low_2 does not work.

Questions (looking again)

As a model theorist, the characterization of models by dimension seems extremely clear. What questions would a computability theorist ask here? And why?

Question

With such a clear understanding of how models work, what is the connection between computing the theory and computing a model? If I know the theory, can I put these ideas to work to produce its models? If I can compute the models, can I reconstruct the theory? And if one model is computable how non-computable can the other models be?

Question

Since all the models look the same, i.e., they are just closures of different sized independent sets, is it true that if one model is computable then other models are computable? If not, which sets of models can be the set of computable models?

Spectrum problem

It follows from the Baldwin-Lachlan theorem that the countable models form an $\omega + 1$-chain: $M_0 \preceq M_1 \preceq \ldots \preceq M_\omega$.

Definition

For a strongly minimal theory T, we let $\mathrm{SRM}(T) = \{\alpha \mid M_\alpha \text{ is computable}\}$.

To formalize our question, we ask:

Question

Which sets $S \subseteq \omega + 1$ can be the Spectrum of Recursive Models of a strongly minimal theory?

All known Examples of Spectra

Answer

The following sets are known to be spectra:

- No models.
- All models.
- $\{0\}$ (Goncharov 1978)
- $[0, n]$ (Kudaibergenov 1980)
- $[0, \omega)$ (Khoussainov, Nies, Shore 1997)
- $[1, \omega]$ (Khoussainov, Nies, Shore 1997)
- $\{1\}$ (Nies 1999)
- $[1, 2, \ldots, n]$ (Nies, Hirschfeldt unpublished)
- $[1, \omega)$ (Nies, Hirschfeldt unpublished)
- $\{\omega\}$ (Hirschfeldt, Khoussainov, Semukhin, 2006)
- $\{0, \omega\}$ (A. 2011)
- $[0, n] \cup \{\omega\}$ (A.-Lempp)
- Any interval (A.)

Upper bound

The only known upper bound for possible spectra is:

Theorem (Nies)

Every spectrum is $\Sigma^0_{\omega+3}$.

Proof.

Suppose T has a computable model (if not, $\emptyset$ is $\Sigma^0_{\omega+3}$), so T and the generic n-type is $\leq_T 0^{(\omega)}$ for each n. Then $n \in \mathrm{SRM}(T)$ iff $\exists$ comp $A \exists c_1 \ldots c_n \in A(A \models p_n(\bar{c})\,\&\,\forall d(\exists \varphi \notin p_{n+1}) A \models \varphi(\bar{c}, d))$ $\square$

This is a big gap, and we seem to be at a loss to provide any better bounds, and new examples are coming quite slowly. Perhaps we can hope for better results for trichotomous theories.

Question

If T is trichotomous, can we use our understanding of the geometry to say something about the spectra?

Regarding disintegrated theories

Theorem (Goncharov-Harizanov-Laskowski-Lempp-McCoy, '03)

If T is disintegrated then $\mathrm{SRM}(T)$ is Σ^0_5.

Theorem (A.-Medvedev, '14)

If T is disintegrated strongly minimal with a finite signature, then $\mathrm{SRM}(T) = \emptyset, [0, \omega]$, or $\{0\}$.

Proof.

Step 1: There is a theory T' which is bi-interpretable with T where every relation symbol in the language of T has rank 1.
Step 2: Algebraic closure in T' can be understood in terms of an easily defined graph relation using these relation symbols.
Step 3: In T', we can use this direct understanding to show $\mathrm{SRM}(T) = \emptyset, [0, \omega]$, or $\{0\}$.
Step 4: Every definable set in a computable disintegrated strongly minimal structure is computable (by GHLLM), so $\mathrm{SRM}(T) = \mathrm{SRM}(T')$. (Note this is non-uniform.) $\square$

The following were...

prepared slides that I did not get to cover during the tutorial series. Since I had these slides prepared (except for this one, obviously), I decided to leave them in for the online version.

In several places, I had intended to draw pictures and give further explanations, but I hope these slides will be of some value anyway.

Disintegrated theories in general

Theorem (A.-Lempp)

The following are exactly the spectra of disintegrated strongly minimal theories with languages are comprised of rank 1 relation symbols: $\emptyset$, $[0, \omega]$, $[1, \omega]$, $\{1\}$, $\{\omega\}$, $[0, n]$, $[0, n] \cup \{\omega\}$, $\{1, \omega\}$[a].

[a] This last one still requires some checking. The main thrust here is that no other set can be such a spectrum

I, perhaps foolishly, believed that if we understood the case for rank 1 languages, and we could understand general ternary languages, then we ought to be able to extend beyond that.

Theorem (A.-Lempp)

There are between 9 and 18 sets which are spectra of strongly minimal disintegrated theories whose languages are comprised of ternary relation symbols.

Despite having mostly figured out the ternary and rank 1 cases, we do not yet know how to attack the general case.

The case of modular groups

We first answer the question for the finite language case.

Theorem (A.-Medvedev, ’14)

If T is a strongly minimal locally modular expansion of a group with a finite signature, then $\mathrm{SRM}(T) = \emptyset$, $[0, \omega]$ or $\{0\}$.

Warning: The next slide is going to say how you can find the quasi-vector space structure of the strongly minimal modular group. Dear non-model-theorist friends, please do not get traumatized; the following slides will be friendly again.

Understanding the interplay here: Quasiendomorphisms

- Let $(G, +, \ldots)$ be a strongly minimal modular group.
- Then $G_0 = \mathrm{acl}_G(\emptyset)$ is a subgroup of G.
- Let Q be the collection of $\mathrm{acl}(\emptyset)$-definable Rank 1 subgroups $K < G \times G$ with projection $\pi_1 : K \to G$ onto.
- Define C to be the collection of quasiendomorphisms of the form $G \times F$ for a finite $F \leq G$.
- Then $D = Q/C$ forms a division ring and G/G_0 is an D-vector-space.
- In fact, $G \cong G/G_0 \oplus G_0$.

Sadly, none of this sounds remotely computable. Why would D be a computable division ring? Why would G_0 be a computable structure?

Recursively Taming Quasiendomorphisms

Lemma

Given $G = (G, +, R_1, \ldots, R_n)$ a modular group, there is $G' = (G, +, H_1, \ldots, H_m)$ where each H_i is a quasiendomorphism so that G' is Δ_1-definable from G, and they have the same quasi-endomorphism ring D and the same $\operatorname{acl}(\emptyset)$. (Note that we don't quite get bi-interpretability).

Draw pictures for geometric idea of reductions
In particular, we give an explicit syntactic translation between any modular group and its presentation as a quasi-vector space structure.

Lemma

If $(G, +, H_1, \ldots H_n)$ is a positive-dimensional modular group where each H_i defines a quasiendomorphism, then the quasiendomorphism ring D is recursively generated by the H_i's. Since $\operatorname{acl}(\emptyset) = \cup_{d \in C} \operatorname{im}(d)$, it is a Σ_1 subset of the universe, thus a recursive structure.

Putting it together

Theorem

If $(G, +, R_1, \ldots R_n)$ is a recursive modular strongly minimal group of positive dimension, then G_0 and D are recursive.

Corollary

If T is a stongly minimal theory of a modular group, then $SRM(T) = \emptyset, \omega + 1,$ or $\{0\}$.

Proof.

If there is a model of positive dimension, then both G_0 and D are recursively presented. From a recursive presentation of D, we can recursively present D^k, the D-vector space of dimension k for any k.

Let G be a model of dimension k $(k \in \omega + 1)$, then $G \cong G_0 \oplus G/G_0 \cong G_0 \oplus D^k$. This gives a recursive presentation of G. □

Infinite language modular groups

The infinite language case is still relatively unexplored. In this case, the division ring is not necessarily finitely generated, and many of our reductions require $0'$. The following is essentially all we know:

Theorem (A.)

For any interval $I \subseteq [0, \omega]$, there is a theory T which is a strongly minimal locally modular expansion of a group so that $\mathrm{SRM}(T) = I$.

Theorem (A.)

If the quasiendomorphism ring is non-commutative or has characteristic 0, then $\mathrm{SRM}(T)$ is computable (in fact, $\mathrm{SRM}(T) \cap [1, \omega)$ is an initial segment of $[1, \omega)$).

Index Sets

On the second slide of the last talk, I claimed that computability theory has something to offer model theory in quantifying the simplest characterizations of things. I had 2 particular theorems in mind, though other examples are not hard to find:

Definition

For P a property of computable thingies[a], we define the index set of P to be the set if i so that the i^{th} computable function is a thingie with property P.

[a] "thingies" could be theories, groups, structures, fields, etc. etc.

Theorem (A.-Makuluni, '13)

The Index set of $\aleph_1$-categorical theories is $\Sigma^0_2 \wedge \Pi^0_2$, and it is complete for this class.

In particular, "ω-stable with no 2-cardinal formula" is distinctly not the simplest characterization of $\aleph_1$-categoricity, since ω-stability is Π^1_1-complete.

A second example: VC-minimality

Definition

A theory T is VC-minimal if there is a set $\Psi = \{\psi_i(x, \bar{y}) \mid i \in I\}$ so that any two instances of formulas from Ψ are either nested or disjoint and so that if A is a definable subset of $M \models T$, then A is a boolean combination of instances of formulas from Ψ.

This is clearly Σ^1_1. This definition seemed like it might be model theoretically useful as a common generalization of o-minimality and strong minimality, but it was incredibly difficult to work with. Indeed, it took non-trivial effort to show that $(\mathbb{Z}, +, <)$ is not VC-minimal.

Theorem (A.-Guingona, '16)

VC-minimality is Π^0_4-complete. In particular, there is a “local” characterization of VC-minimality for countable theories.

The end

Thank you!