Computable Model Theory Part 1:

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Because this is the only setting that I know where computable/recursive and decidable do not mean the same thing, I want to start with clarifying:

Definition

A structure M is computable if it has domain (a computable subset of) ω and its atomic diagram is a computable set (in particular, we assume the language is computable as well). Example: $(\mathbb{N}, +, \cdot)$

Definition

A structure M is decidable if it has domain (a computable subset of) ω and its elementary diagram is a computable set. Example: $(\mathbb{Q}, <)$.

Model Theory and Computability are kin

Computability theory looks at things in a level of detail and a different perspective than Model theorists are used to. This perspective interplays well with model theory:

- By asking questions that otherwise would not arise, which gives rise to new model theoretic ideas.
- By quantifying and clarifying ideas existent already in model theory. For example: Is this really the simplest characterization of this property, or is there an equivalent local definition?

Historically, this was the case. Tarski did not prove "quantifier elimination" of $\mathbb{R}$. He developed that idea in order to prove "decidability". To this day, decidability is considered a litmus test for having mastered (the first order content of) a theory.

Plan for these talks

I plan to present several examples of an exchange of ideas between computability and model theory.

I will also hope to present several topics which I believe both model theorists and computability theorists can find interesting.

I am also very excited to share with you a theorem of Marker's which is my favorite theorem (it appeals deeply to my personal sense of beauty – and is a remarkable example of the above theme).

Most of all, my goal is for you to consider some questions or ideas from computable model theory as interesting, and I hope to help begin a few conversations.

Much of my own work has looked at computability theoretic questions about strongly minimal theories, so I will start there.

Definition

A first order theory T is *strongly minimal* if every definable subset of every model is a finite or co-finite subset of the model.

Example

- A regular acyclic graph with finite valence (say, the theory of a Cayley graph of a finitely generated group);
- A vector space (say, the theory of $(\mathbb{Q}, +)$);
- An algebraically closed field, (say, the theory of $(\mathbb{C}, +, \cdot, 0, 1)$)

Algebraic closure and independence

Definition

For elements $\bar{a}, b \in M$, we say $b \in \operatorname{acl}(\bar{a})$ if there is a formula $\varphi(x, \bar{y})$ so that $\varphi(M, \bar{a})$ is finite and $M \models \varphi(b, \bar{a})$.
i.e., b is in a finite $\bar{a}$-definable set.

Definition

A set $S \subset M$ is independent if each $x \in S$ is not in $\operatorname{acl}(S \setminus \{x\})$.

Definition

For any n, there is a unique type of an independent n-tuple. We call this the generic n-type.

Dimension

Definition

- If M is a model, a maximal independent subset is called a basis for M.
- The dimension of a set X is the size of a maximal independent subset.

Fact

In strongly minimal theories, the above are well-defined. That is, if B_1 and B_2 are maximal independent subsets of X, then $|B_1| = |B_2|$.

Morley rank

Definition

Let $\varphi(\bar{x})$ be a formula and T a strongly minimal theory. Then the *Morley rank* of $\varphi(\bar{x})$ is the maximal dimension of a tuple $\bar{a} \in M \models T$ so that $M \models \varphi(\bar{a})$.

Examples

Consider the formula $x + x = y$ in the theory of $\mathbb{Q}$-vector spaces.
This formula has Morley rank 1.
The formula $x + y = z$ has Morley rank 2.

Dimension characterizes models

In each of our examples, the notion of dimension characterizes models. This is not a coincidence.

Theorem (Baldwin-Lachlan)

If T is strongly minimal[a], then each model of T is determined by its dimension. If M is countable, then $\dim(M) \in \{0, 1, \ldots, \aleph_0\}$.

[a] They actually showed the result for $\aleph_1$-categorical theories, but I will talk only about strongly minimal theories

Zilber conjectured that in fact our canonical examples of strongly minimal theories formed an exhaustive list.

Zilber conjectured that every strongly minimal theory was of one of three types:

- Disintegrated (Essentially binary)
- Locally Modular (Essentially a quasi-vector space)
- Field-like (Essentially an algebraically closed field)

Theorem (Hrushovski 1991)

The Zilber trichotomy is false. There are Hrushovski constructions which build non-trichotomous theories.

These structures are inherently combinatorial in nature, and have no algebraic content.

Questions

As a model theorist, the characterization of models by dimension seems extremely clear. What questions would a computability theorist ask here? And why?

Question

With such a clear understanding of how models work, what is the connection between computing the theory and computing a model? If I know the theory, can I put these ideas to work to produce its models? If I can compute the models, can I reconstruct the theory? And if one model is computable how non-computable can the other models be?

Question

Since all the models look the same, i.e., they are just closures of different sized independent sets, is it true that if one model is computable then other models are computable? If not, which sets of models can be the set of computable models?

More questions

Question

To each of the questions above, how does the geometry of the theory (i.e., whether it falls into the Zilber trichotomy and if so, under which category) affect the answer? Is everything as computable as can be for disintegrated theories? Modular theories?

Question

If we restrict the language to a finite signature, does that change anything?

This last may seem a strange question to ask, but will be better motivated after seeing some of the examples built using infinite signatures, and some of the theorems we can prove for theories in finite signatures.

Theories compute models

To answer the first direction of the first question:

Theorem (Harrington, Khisamiev '74)

If T is computable and strongly minimal, then every countable model of T is decidable.

Proof.

By definability of Morley rank in a strongly minimal theory, there is an algorithm (given that you can compute T) for extending a formula to an isolated type: Consider each formula in turn and choose to add ψ to the finite partial type p_0 if $MR(p_0 \cup \{\psi\}) < MR(p_0)$.
The proof of the theorem that a countable atomic theory has a prime model along with a finite injury construction to make all the types assigned consistent with each other, shows that this suffices for the prime model to be decidable.
Lastly the unique generic k-type p_k is computable from T for $k \in \omega + 1$. Repeat with p_k in place of T. □

Models do not compute theories... at all

In general, if M is computable, then $\mathrm{Th}(M)$ is in some turing degree $\leq_T 0^{(\omega)}$ (where $0^{(\omega)}$ is the full theory of first order arithmetic). The canonical example of this and (by most metrics) the worst possible computable structure is $(\mathbb{N}, +, \cdot)$.

Theorem (A., 2013)

There is a strongly minimal theory T so that $T \equiv_T 0^{(\omega)}$ and every countable model of T is computable.

This proof requires, essentially "nesting" infinitely many levels of a Hrushovski construction. It led to new ideas in the use of Hrushovski constructions, which most likely would not have arisen without the question coming from computability. We will see below hints towards showing that this T cannot be trichotomous.

The disintegrated case

Theorem (Goncharov-Harizanov-Laskowski-Lempp-McCoy, '03)

If T is a disintegrated strongly minimal theory and $M \models T$, then the elementary diagram of M is model complete.

This statement is pure model theory, but it came about only to prove the following. A hint at the connection: Model complete theories are $\forall\exists$-axiomatizable.

Theorem (Goncharov-Harizanov-Laskowski-Lempp-McCoy, '03)

If T is a disintegrated strongly minimal theory and it has one computable model, then T is computable from $0''$.

Corollary

If T is a disintegrated strongly minimal theory and it has one computable model, then every countable model of T is computable in $0''$.

But this (model theoretic) line of research did not end there:

Mutually algebraic structures and model completeness

Theorem (Dolich-Laskowski-Raichev, '06)

If T is $\aleph_1$-categorical, disintegrated, has Morley rank 1, and $M \models T$, then the elementary diagram of M is model complete.

How about rank 2? We might expect quantifier elimination down to $\exists\forall$ formulas instead of model completeness.

Theorem (Laskowski, '09)

If M is a model of a disintegrated, weakly minimal theory, then the elementary diagram of M is almost model complete, i.e., every formula is equivalent to a boolean combination of existential "mutually algebraic formulas".

[Laskowski, '15] also gave a characterization for which "mutually algebraic structures" (where every formula is equivalent to a boolean combination of mutually algebraic formulas) it is true that the elementary diagram is model complete. He uses this result to give cleaner proofs of the DLR and GHLLM results.

The locally modular case

Theorem (A.-Medvedev, '14)

If T is a strongly minimal locally modular theory expanding a group and $M \models T$, then the elementary diagram of M is almost model complete.

Corollary (Of the proof)

If T is a strongly minimal modular theory expanding a group and has one computable model, then T is computable in $0''$. Further, if T has a computable positive-dimensional model, then T is computable in $0'$.

Question

How about general locally modular theories?

Observation

If T is a locally modular strongly minimal theory, not only is there a pure group interpretable in T, but it is Δ^0_1-interpretable.

The end

Thank you!